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ADDITIVITY OF THE GERLITS–NAGY PROPERTY AND CONCENTRATED SETS

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ABSTRACT. We settle all problems concerning the additivity of the Gerlits–Nagy property and related additivity numbers posed by Scheepers in his tribute paper to Gerlits. We apply these results to compute the minimal number of concentrated sets of reals (in the sense of Besicovitch) whose union, when multiplied with a Gerlits–Nagy space, need not have Rothberger's property. We apply these methods to construct a large family of spaces whose product with every Hurewicz space has Menger's property. Our applications extend earlier results of Babinkostova and Scheepers.

1. INTRODUCTION

We consider the preservation of several classic topological properties under unions. These properties are best understood in the broader context of topological selection principles. We thus provide, in the present section, a brief introduction.[1] This framework was introduced by Scheepers in [14] to study, in a uniform manner, a variety of properties introduced in different mathematical disciplines, since the early 1920s, by Menger, Hurewicz, Rothberger, Gerlits and Nagy, and many others.

By *space* we mean an infinite topological space. Let X be a space. We say that $\mathcal{U}$ is a *cover* of X if $X = \bigcup\mathcal{U}$, but $X \notin \mathcal{U}$. Often, X is considered as a subspace of another space Y, and in this case we always consider covers of X by subsets of Y and require instead that no member of the cover contains X. Let $\mathrm{O}(X)$ be the family of all countable open covers of X.[2] Define the following subfamilies of $\mathrm{O}(X)$: $\mathcal{U} \in \Omega(X)$ if each finite subset of X is contained in some member of $\mathcal{U}$, $\mathcal{U} \in \Gamma(X)$ if $\mathcal{U}$ is infinite, and each element of X is contained in all but finitely many members of $\mathcal{U}$.

Some of the following statements may hold for families $\mathscr{A}$ and $\mathscr{B}$ of covers of X:

- $\binom{\mathscr{A}}{\mathscr{B}}$**:** Each member of $\mathscr{A}$ contains a member of $\mathscr{B}$.
- $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$**:** For each sequence $\langle \mathcal{U}_n \in \mathscr{A} : n \in \mathbb{N}\rangle$, there is a selection $\langle U_n \in \mathcal{U}_n : n \in \mathbb{N}\rangle$ such that $\{U_n : n \in \mathbb{N}\} \in \mathscr{B}$.

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[1]This introduction is adopted from [11]. Extended introductions to this field are available in [10, 15, 17].

[2]Our assumption that the considered covers are countable may be replaced by assuming that all considered spaces are Lindelöf in all finite powers, e.g., subsets of the real line.

$\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$**:** For each sequence $\langle \mathcal{U}_n \in \mathscr{A} : n \in \mathbb{N}\rangle$, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n : n \in \mathbb{N}\rangle$ such that $\bigcup_n \mathcal{F}_n \in \mathscr{B}$.

$\mathsf{U}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$**:** For each sequence $\langle \mathcal{U}_n \in \mathscr{A} : n \in \mathbb{N}\rangle$, where no $\mathcal{U}_n$ contains a finite subcover, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n : n \in \mathbb{N}\rangle$ such that $\{\bigcup \mathcal{F}_n : n \in \mathbb{N}\} \in \mathscr{B}$.

We say, e.g., that X satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ if the statement $\mathsf{S}_1(\mathrm{O}(X),\mathrm{O}(X))$ holds. This way, $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ is a property (or a class) of spaces, and similarly for all other statements and families of covers. Each nontrivial property among these properties, where $\mathscr{A},\mathscr{B}$ range over O,Ω,Γ, is equivalent to one in Figure 1 [8, 14]. In this diagram, an arrow denotes implication.

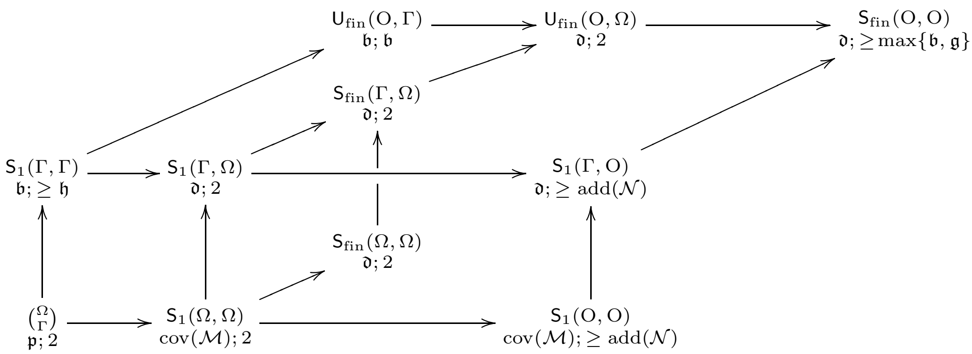

FIGURE 1. The Scheepers Diagram

The extremal properties in the Scheepers Diagram—$\binom{\Omega}{\Gamma}$, $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$, and $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$—where introduced (sometimes in another, equivalent form) in the classic works of Gerlits and Nagy, Rothberger, Hurewicz, and Menger, respectively. In addition, we indicate below each class P its *critical cardinality* $\mathrm{non}(P)$ (the minimal cardinality of a space not in the class), followed by its *additivity number* $\mathrm{add}(P)$ (the minimal number of spaces in the class with union outside the class). When only upper and lower bounds are known, we write a lower bound. To save space, we do not write the immediate upper bound, $\mathrm{cf}(\mathrm{non}(P))$. These cardinals are all combinatorial cardinal characteristics of the continuum, details about which are available in [5]. Here, $\mathcal{M},\mathcal{N}$ are the families of meager sets in $\mathbb{R}$ and Lebesgue null sets in $\mathbb{R}$, respectively. Complete computations of the mentioned additivity numbers and bounds, with references, are available in [18]. That the additivity number of $\mathsf{S}_1(\Gamma,\mathrm{O})$ is $\geq \mathrm{add}(\mathcal{N})$ follows from Bartoszyński's Theorem [18, Lemma 2.16] and the first observation in [19, Appendix A].

Many additional—classic and new—properties are studied in relation to the Scheepers Diagram. One of these is the Gerlits–Nagy property, to which we now turn our attention.

2. Additivity of the Gerlits–Nagy property

Definition 2.1. For classes P, Q of spaces, $\operatorname{add}(P,Q)$ is the minimal cardinal κ such that some union of κ members of P is not in Q. $\operatorname{add}(P)$ is $\operatorname{add}(P,P)$.

A countable cover $\mathcal{U}$ of a space X is in $\gimel(\Gamma)$ ($\gimel$, read *gimel*, for brevity)[3] if for each (equivalently, some) bijective enumeration $\mathcal{U} = \{U_n : n \in \mathbb{N}\}$ there is an increasing $h \in \mathbb{N}^{\mathbb{N}}$ such that, for each $x \in X$,

$$x \in \bigcup_{k=h(n)}^{h(n+1)-1} U_k$$

for all but finitely many n.

The property $\mathsf{S}_1(\Omega, \gimel)$ was introduced, in an equivalent form, by Gerlits and Nagy in their seminal paper [7]. Building on results of Gerlits and Nagy and extending them, Kočinac and Scheepers prove in [9] that

$$\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma) \cap \mathsf{S}_1(\mathrm{O}, \mathrm{O}) = \mathsf{S}_1(\Omega, \gimel).$$

This property is often referred to as the *Gerlits–Nagy property* [16].

The importance of the Gerlits–Nagy property $\mathsf{S}_1(\Omega, \gimel)$ in various contexts is surveyed in Scheepers's tribute to Gerlits [16]. In [16, § II.5], Scheepers poses several problems concerning preservation of this property under unions. All these problems of Scheepers are settled by the following two theorems.

Theorem 2.2.

$$\operatorname{add}(\mathsf{S}_1(\Omega, \gimel), \mathsf{S}_1(\mathrm{O}, \mathrm{O})) = \operatorname{add}(\tbinom{\Omega}{\Gamma}, \mathsf{S}_1(\mathrm{O}, \mathrm{O})) = \operatorname{cov}(\mathcal{M}).$$

Proof. Since $\binom{\Omega}{\Gamma} = \mathsf{S}_1(\Omega, \Gamma)$ [7] implies $\mathsf{S}_1(\Omega, \gimel)$,

$$\operatorname{add}(\mathsf{S}_1(\Omega, \gimel), \mathsf{S}_1(\mathrm{O}, \mathrm{O})) \leq \operatorname{add}(\tbinom{\Omega}{\Gamma}, \mathsf{S}_1(\mathrm{O}, \mathrm{O})) \leq \operatorname{non}(\mathsf{S}_1(\mathrm{O}, \mathrm{O})) = \operatorname{cov}(\mathcal{M}).$$

It remains to prove that $\operatorname{cov}(\mathcal{M}) \leq \operatorname{add}(\mathsf{S}_1(\Omega, \gimel), \mathsf{S}_1(\mathrm{O}, \mathrm{O}))$. We use the fact that $\mathsf{S}_1(\mathrm{O}, \mathrm{O}) = \mathsf{S}_1(\Omega, \mathrm{O})$ [14].

Let $\kappa < \operatorname{cov}(\mathcal{M})$. Assume that, for each $\alpha < \kappa$, X_α satisfies $\mathsf{S}_1(\Omega, \gimel)$, and $X = \bigcup_{\alpha<\kappa} X_\alpha$. Let $\mathcal{U}_n \in \Omega(X)$ for all n. Enumerate $\mathcal{U}_n = \{U^n_m : m \in \mathbb{N}\}$. For each α, as X_α satisfies $\mathsf{S}_1(\Omega, \gimel)$, there are $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ and an increasing $h_\alpha \in \mathbb{N}^{\mathbb{N}}$ such that, for each $x \in X_\alpha$,

$$x \in \bigcup_{k=h_\alpha(n)}^{h_\alpha(n+1)-1} U^k_{f_\alpha(k)}$$

for all but finitely many n.

Since $\kappa < \operatorname{cov}(\mathcal{M}) \leq \mathfrak{d}$ [5], there is an increasing $h \in \mathbb{N}^{\mathbb{N}}$ such that, for each $\alpha < \kappa$, the set

$$I_\alpha = \{n : [h_\alpha(n), h_\alpha(n+1)) \subseteq [h(n), h(n+1))\}$$

is infinite [5]. For each $\alpha < \kappa$, define

$$g_\alpha \in \prod_{n \in I_\alpha} \mathbb{N}^{[h(n), h(n+1))}$$

[3] In general, the gimel operator $\gimel$ can be applied to any type of cover [13]. However, in the present paper we apply it only to Γ, and omit the symbol Γ for brevity.

by $g_\alpha(n) = f_\alpha \restriction [h(n), h(n+1))$ for all $n \in I_\alpha$. As $\kappa < \operatorname{cov}(\mathcal{M})$, by Lemma 2.4.2(3) in [2], there is $g \in \prod_n \mathbb{N}^{[h(n),h(n+1))}$ guessing all functions g_α; that is, for each $\alpha < \kappa$, $g(n) = g_\alpha(n)$ for infinitely many $n \in I_\alpha$ [5]. Define $f \in \mathbb{N}^\mathbb{N}$ by $f(k) = g(n)(k)$, where n is the one with $k \in [h(n), h(n+1))$. Then $\{U^n_{f(n)} : n \in \mathbb{N}\} \in \mathrm{O}(X)$.

Indeed, let $x \in X$. Pick $\alpha < \kappa$ with $x \in X_\alpha$. Pick m such that, for all $n > m$, $x \in \bigcup_{k=h_\alpha(n)}^{h_\alpha(n+1)-1} U^k_{f_\alpha(k)}$. Pick $n \in I_\alpha$ such that $n > m$ and $g(n) = g_\alpha(n)$. Then

$$x \in \bigcup_{k=h_\alpha(n)}^{h_\alpha(n+1)-1} U^k_{f_\alpha(k)} \subseteq \bigcup_{k=h(n)}^{h(n+1)-1} U^k_{f_\alpha(k)} = \bigcup_{k=h(n)}^{h(n+1)-1} U^k_{f(k)}. \qquad \square$$

We can now compute the additivity number of the Gerlits–Nagy property.

Theorem 2.3. $\operatorname{add}(\mathsf{S}_1(\Omega, \beth)) = \operatorname{add}(\mathcal{M})$.

Proof. As $\mathsf{S}_1(\Omega, \beth) = \mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma) \cap \mathsf{S}_1(\mathrm{O}, \mathrm{O})$,

$$\begin{aligned} \operatorname{add}(\mathsf{S}_1(\Omega, \beth)) &\leq \operatorname{non}(\mathsf{S}_1(\Omega, \beth)) \\ &= \min\{\operatorname{non}(\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)), \operatorname{non}(\mathsf{S}_1(\mathrm{O}, \mathrm{O}))\} \\ &= \min\{\mathfrak{b}, \operatorname{cov}(\mathcal{M})\} = \operatorname{add}(\mathcal{M}). \end{aligned}$$

It remains to prove the other inequality. Let $X = \bigcup_{\alpha<\kappa} X_\alpha$, with each X_α in $\mathsf{S}_1(\Omega, \beth)$, and $\alpha < \operatorname{add}(\mathcal{M})$. By Theorem 2.2, X satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. As $\kappa < \mathfrak{b} = \operatorname{add}(\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma))$ [18], X satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$, too. Thus, X satisfies $\mathsf{S}_1(\Omega, \beth)$. $\square$

The following definition and corollary will be used in the next section.

Definition 2.4. Let P, Q be classes of spaces, each containing all one-element spaces and closed under homeomorphic images. $\big(P, Q\big)^\times$ is the class of all spaces X such that, for each Y in P, $X \times Y$ is in Q. $\big(P, P\big)^\times$ is denoted $P^\times$.

Lemma 2.5. *Let P, Q be classes of spaces. Then:*

(1) $\operatorname{add}(P, Q) \leq \operatorname{non}(\big(P, Q\big)^\times) \leq \operatorname{non}(Q)$.
(2) $\operatorname{add}(Q) \leq \operatorname{add}(\big(P, Q\big)^\times) \leq \operatorname{non}(Q)$. $\square$

Corollary 2.6.

$$\operatorname{non}(\big(\mathsf{S}_1(\Omega, \beth), \mathsf{S}_1(\mathrm{O}, \mathrm{O})\big)^\times) = \operatorname{non}(\big(\tbinom{\Omega}{\Gamma}, \mathsf{S}_1(\mathrm{O}, \mathrm{O})\big)^\times) = \operatorname{cov}(\mathcal{M}).$$

Proof. Theorem 2.2 and Lemma 2.5. $\square$

Remark 2.7. The above proofs, verbatim, show that the results of the present section also apply in the case where countable *Borel* covers are considered instead of countable open covers.

3. Unions of concentrated sets

According to Besicovitch [3,4], a space X is *concentrated* if there is a countable $D \subseteq X$ such that for each open $U \supseteq D$, $X \setminus U$ is countable. More generally, for a cardinal κ, a space X is *κ-concentrated* if there is a countable $D \subseteq X$ such that for each open $U \supseteq D$, $|X \setminus U| < \kappa$. The classic examples of concentrated spaces are Luzin sets. Modern examples are constructed from scales, following and extending methods of Rothberger (e.g., [20]).

Babinkostova and Scheepers proved in [1] that every concentrated metric space belongs to $\left(\mathsf{S}_1(\Omega,\daleth),\mathsf{S}_1(\mathrm{O},\mathrm{O})\right)^{\times}$.[4] In other words, for each concentrated metric space C, if Y satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, then $C\times Y$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. We generalize this result in several ways.

Theorem 3.1.

(1) *Let λ be a regular uncountable cardinal $\leq \mathrm{cov}(\mathcal{M})$. The minimal number of λ-concentrated spaces, whose union is a regular space not satisfying $\left(\mathsf{S}_1(\Omega,\daleth),\mathsf{S}_1(\mathrm{O},\mathrm{O})\right)^{\times}$, is $\mathrm{cov}(\mathcal{M})$.*

(2) *The minimal number of $\mathrm{cov}(\mathcal{M})$-concentrated spaces, whose union is a regular space not satisfying $\left(\mathsf{S}_1(\Omega,\daleth),\mathsf{S}_1(\mathrm{O},\mathrm{O})\right)^{\times}$, is $\mathrm{cf}(\mathrm{cov}(\mathcal{M}))$.*

Proof. We prove both statements simultaneously.

There is a set of real numbers, of cardinality $\mathrm{cov}(\mathcal{M})$, that does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ [8]. Thus, the minimal number sought after is at most $\mathrm{cov}(\mathcal{M})$ for (1) and at most $\mathrm{cf}(\mathrm{cov}(\mathcal{M}))$ for (2).

Let λ be a regular cardinal $\leq \mathrm{cov}(\mathcal{M})$ for (1), and $\mathrm{cov}(\mathcal{M})$ for (2). Let $\kappa < \mathrm{cov}(\mathcal{M})$ for (1), and $< \mathrm{cf}(\mathrm{cov}(\mathcal{M}))$ for (2).

Let $C=\bigcup_{\alpha<\kappa}C_\alpha$ be a regular space, with each C_α λ-concentrated on some countable set $D_\alpha\subseteq C_\alpha$. Let Y be a space satisfying $\mathsf{S}_1(\Omega,\daleth)$. We must prove that $C\times Y$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.

Let K be a compact space containing C as a subspace. Let $\mathcal{U}_n$, $n\in\mathbb{N}$, be countable covers of $C\times Y$ by sets open in $K\times Y$. Let $D=\bigcup_{\alpha<\kappa}D_\alpha$. As $|D|=\kappa<\mathrm{cov}(\mathcal{M})$, we have by Corollary 2.6 that $D\times Y$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. Thus, pick $U_n\in\mathcal{U}_n$, $n\in\mathbb{N}$, such that $D\times Y\subseteq U:=\bigcup_n U_n$.

The Hurewicz property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ is preserved by products with compact spaces, moving to closed subspaces, and continuous images [8]. Since Y satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and K is compact, $K\times Y$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. Thus, so does $K\times Y\setminus U$. It follows that the projection H of $K\times Y\setminus U$ on the first coordinate satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. Note that

$$(K\setminus H)\times Y\subseteq U.$$

The argument in the proof of [8, Theorem 5.7] generalizes to regular spaces, to show that for H,F disjoint subspaces of a regular space K with H $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$, and F F_σ, there is a G_δ set $G\subseteq K$ such that $G\supseteq F$ and $H\cap G=\emptyset$.

For each $\alpha<\kappa$, let G_α be a G_δ subset of K such that $D_\alpha\subseteq G_\alpha$ and $H\cap G_\alpha=\emptyset$. As C_α is λ-concentrated on D_α, $C_\alpha\setminus G_\alpha$ is a countable union of sets of cardinality $<\lambda$.

As λ has uncountable cofinality, $|C_\alpha\setminus G_\alpha|<\lambda$. Then

$$C\cap H\subseteq C\setminus\bigcup_{\alpha<\kappa}G_\alpha\subseteq\bigcup_{\alpha<\kappa}C_\alpha\setminus G_\alpha.$$

By splitting to cases $\lambda<\mathrm{cov}(\mathcal{M})$ and $\lambda=\mathrm{cov}(\mathcal{M})$, one sees that $|C\cap H|<\mathrm{cov}(\mathcal{M})$ in both scenarios (1) and (2). Thus, by Corollary 2.6 again, $(C\cap H)\times Y$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, and there are $V_n\in\mathcal{U}_n$, $n\in\mathbb{N}$, such that $(C\cap H)\times Y\subseteq\bigcup_n V_n$. In summary,

$$C\times Y\subseteq((K\setminus H)\times Y)\cup((C\cap H)\times Y)\subseteq\bigcup_{n\in\mathbb{N}}(U_n\cup V_n).$$

[4]This is a special case of their Theorem 11(3). Their general result will be explained and generalized further in Theorem 3.3 and the discussion preceding it.

We have picked two sets (instead of one) from each cover $\mathcal{U}_n$, but this is fine [6] (cf. [19, Appendix A]). $\square$

Definition 3.2. Let κ be an infinite cardinal number. Let $\mathscr{C}_0(\kappa)$ be the family of regular spaces of cardinality $< \kappa$. For successor ordinals $\alpha+1$, let $C \in \mathscr{C}_{\alpha+1}(\kappa)$ if C is regular, and:

(1) either there is a countable $D \subseteq C$ with $C \setminus U \in \mathscr{C}_\alpha(\kappa)$ for all open $U \supseteq D$
(2) or C is a union of less than $\operatorname{cf}(\kappa)$ members of $\mathscr{C}_\alpha(\kappa)$.

For limit ordinals α, let $\mathscr{C}_\alpha(\kappa) = \bigcup_{\beta<\alpha} \mathscr{C}_\beta(\kappa)$.

Modulo a metrizability assumption, the Babinkostova–Scheepers Theorem [1, Theorem 11(3)] asserts that that every member of $\mathscr{C}_{\aleph_0}(2)$ is in $\big(\mathsf{S}_1(\Omega,\gimel),\mathsf{S}_1(\mathrm{O},\mathrm{O})\big)^\times$. We use our methods to prove the following, stronger result.

For the following theorem, we recall from the Scheepers Diagram that

$$\operatorname{add}(\mathcal{N}) \le \operatorname{add}(\mathsf{S}_1(\mathrm{O},\mathrm{O})) \le \operatorname{cf}(\operatorname{cov}(\mathcal{M})).$$

Theorem 3.3. *The product of each member of $\mathscr{C}_{\operatorname{add}(\mathcal{N})}(\operatorname{cov}(\mathcal{M}))$ with every member of $\mathsf{S}_1(\Omega,\gimel)$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.*

Proof. We prove the stronger assertion, with $\operatorname{add}(\mathsf{S}_1(\mathrm{O},\mathrm{O}))$ instead of $\operatorname{add}(\mathcal{N})$.

For brevity, let $\mathscr{C}_\alpha := \mathscr{C}_\alpha(\operatorname{cov}(\mathcal{M}))$ for all α. By induction on $\alpha \le \operatorname{add}(\mathsf{S}_1(\mathrm{O},\mathrm{O}))$, we prove that

$$\mathscr{C}_\alpha \subseteq \big(\mathsf{S}_1(\Omega,\gimel),\mathsf{S}_1(\mathrm{O},\mathrm{O})\big)^\times.$$

The proof is similar to that of Theorem 3.1, so we omit some of the explanations.

The case $\alpha = 0$ is treated in Corollary 2.6. For limit α, there is nothing to prove.

$\alpha+1$: Let $C \in \mathscr{C}_{\alpha+1}$. Let K be a compact space containing C as a subspace. Let Y be a space satisfying $\mathsf{S}_1(\Omega,\gimel)$.

First case: There is a countable $D \subseteq C$ with $C \setminus U \in \mathscr{C}_\alpha$ for all open $U \supseteq D$. Given $\mathcal{U}_n \in \mathrm{O}(C \times Y)$, pick $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $D \times Y \subseteq U := \bigcup_n U_n$. Let H be the projection of $K \times Y \setminus U$ on the first coordinate. Let G be a G_δ subset of K such that $D \subseteq G$ and $H \cap G = \emptyset$. $C \setminus G$ is a countable union of elements of $\mathscr{C}_\alpha$. By the induction hypothesis and Corollary 2.6, $C \setminus G \in \big(\mathsf{S}_1(\Omega,\gimel),\mathsf{S}_1(\mathrm{O},\mathrm{O})\big)^\times$. Then $(C \setminus G) \times Y$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, and there are $V_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $(C \cap H) \times Y \subseteq (C \setminus G) \times Y \subseteq \bigcup_n V_n$. In summary,

$$C \times Y \subseteq ((K \setminus H) \times Y) \cup ((C \cap H) \times Y) \subseteq \bigcup_{n\in\mathbb{N}} (U_n \cup V_n).$$

Second case: There are $\kappa < \operatorname{cf}(\operatorname{cov}(\mathcal{M}))$ and $C_\beta \in \mathscr{C}_\alpha$, $\beta < \kappa$ such that $C = \bigcup_{\beta<\kappa} C_\beta$. For each $\beta < \kappa$ with C_β a union of less than $\operatorname{cf}(\operatorname{cov}(\mathcal{M}))$ members of

$$\mathscr{C}_{<\alpha} := \bigcup_{\gamma<\alpha} \mathscr{C}_\gamma,$$

we may take all elements in all of these unions instead of the original C_β's. Thus, we may assume that for each C_β there is a countable (possibly empty) $D_\beta \subseteq C_\beta$ with

$$C_\beta \setminus U \in \mathscr{C}_{<\alpha}$$

for all open $U \supseteq D_\beta$. Let $D = \bigcup_{\beta<\kappa} D_\beta$. Then $|D| < \operatorname{cov}(\mathcal{M})$.

Given $\mathcal{U}_n \in \mathrm{O}(C \times Y)$, pick $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $D \times Y \subseteq U := \bigcup_n U_n$. Let H be the projection of $K \times Y \setminus U$ on the first coordinate. For each $\beta < \kappa$, let G_β be a G_δ subset of K such that $D_\beta \subseteq G_\beta$ and $H \cap G_\beta = \emptyset$. Let $G = \bigcup_{\beta<\kappa} G_\beta$. Now,

$$C \cap H \subseteq \bigcup_{\beta<\kappa} C_\beta \setminus G_\beta,$$

where each $C_\beta \setminus G_\beta$ is a countable union of elements of $\mathscr{C}_{<\alpha}$. All in all, we arrive at a union of a family $\mathcal{F} \subseteq \mathscr{C}_{<\alpha}$ with $|\mathcal{F}| < \mathrm{cf}(\mathrm{cov}(\mathcal{M}))$, and we must show that $\bigcup \mathcal{F} \in \big(\mathsf{S}_1(\Omega, \daleth), \mathsf{S}_1(\mathrm{O},\mathrm{O})\big)^\times$. Indeed, for each $\gamma < \alpha$,

$$X_\gamma := \bigcup (\mathcal{F} \cap \mathscr{C}_\gamma) \in \mathscr{C}_{\gamma+1} \subseteq \mathscr{C}_\alpha.$$

By the induction hypothesis, $X_\gamma \in \big(\mathsf{S}_1(\Omega, \daleth), \mathsf{S}_1(\mathrm{O},\mathrm{O})\big)^\times$. By Corollary 2.6, since $\alpha < \mathrm{add}(\mathsf{S}_1(\mathrm{O},\mathrm{O}))$,

$$\bigcup \mathcal{F} = \bigcup_{\gamma<\alpha} X_\gamma \in \big(\mathsf{S}_1(\Omega, \daleth), \mathsf{S}_1(\mathrm{O},\mathrm{O})\big)^\times.$$

It follows that $(C \cap H) \times Y \subseteq (C \setminus G) \times Y \subseteq \bigcup_n V_n$ for some $V_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, and $C \times Y \subseteq \bigcup_{n\in\mathbb{N}} (U_n \cup V_n)$. □

4. Spaces whose product with Hurewicz spaces are Menger

A space is *σ-compact* if it is a union of countably many compact spaces.

Definition 4.1. For a cardinal λ, K_λ is the family of all spaces that are unions of less than λ compact spaces. A space X is *K_λ-concentreted* if there is a σ-compact subset $D \subseteq X$ such that $X \setminus U \in K_\lambda$ for each open $U \supseteq D$.

Babinkostova and Scheepers proved in [1, Theorem 11(2)] that, for each concentrated metric space C, if Y has Hurewicz's property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, then $C \times Y$ has Menger's property $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.[5] We use the methods of the previous section to generalize this result. Since the proofs are almost a literal repetition of the corresponding ones in the previous section, we omit some of the details.

The following is immediate from the definitions.

Lemma 4.2 (Folklore)**.** $\mathrm{add}(\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma), \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})) = \mathfrak{d}$.

Lemma 4.3. $K_{\mathfrak{d}} \subseteq \big(\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma), \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})\big)^\times$.

Proof. Each compact space is in $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)^\times$. Apply Lemma 4.2. □

Theorem 4.4.

(1) *Let λ be a regular uncountable cardinal $\leq \mathfrak{d}$. The minimal number of K_λ-concentrated spaces, whose union is a regular space not satisfying $\big(\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma), \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})\big)^\times$, is $\mathfrak{d}$.*

(2) *The minimal number of $K_{\mathfrak{d}}$-concentrated spaces, whose union is a regular space not satisfying $\big(\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma), \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})\big)^\times$, is $\mathrm{cf}(\mathfrak{d})$.*

[5] Here, too, the Babinkostova–Scheepers Theorem is more general. Their full result is generalized in our forthcoming Theorem 4.6.

Proof. There is a set of real numbers of cardinality $\mathfrak{d}$ that does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ [8]. Thus, the minimal number sought after is at most $\mathfrak{d}$ for (1) and at most $\mathrm{cf}(\mathfrak{d})$ for (2).

Let λ be a regular cardinal $\leq \mathfrak{d}$ for (1), and $\mathfrak{d}$ for (2). Let $\kappa < \mathfrak{d}$ for (1), and $< \mathrm{cf}(\mathfrak{d})$ for (2).

Let $C = \bigcup_{\alpha<\kappa} C_\alpha$ be a regular space, with each C_α K_λ-concentrated on some σ-compact set $D_\alpha \subseteq C_\alpha$. Let Y be a space satisfying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. We must prove that $C \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.

Let K be a compact space containing C as a subspace. Let $\mathcal{U}_n$, $n \in \mathbb{N}$, be countable covers of $C \times Y$ by sets open in $K \times Y$. Let $D = \bigcup_{\alpha<\kappa} D_\alpha$. As $D \in K_{\mathfrak{d}}$, we have by Lemma 4.3 that $D \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. Thus, pick finite $\mathcal{F}_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $D \times Y \subseteq U := \bigcup_n \bigcup \mathcal{F}_n$.

Since Y satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and K is compact, the projection H of $K \times Y \setminus U$ on the first coordinate satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. Note that

$$(K \setminus H) \times Y \subseteq U.$$

For each $\alpha < \kappa$, let G_α be a G_δ subset of K such that $D_\alpha \subseteq G_\alpha$ and $H \cap G_\alpha = \emptyset$. As C_α is K_λ-concentrated on D_α, $C_\alpha \setminus G_\alpha$ is a countable union of elements of K_λ.

As λ has uncountable cofinality, $C_\alpha \setminus G_\alpha \in K_\lambda$. Then

$$C \cap H \subseteq C \setminus \bigcup_{\alpha<\kappa} G_\alpha \subseteq \tilde{C} := \bigcup_{\alpha<\kappa} C_\alpha \setminus G_\alpha.$$

By splitting to cases $\lambda < \mathfrak{d}$ and $\lambda = \mathfrak{d}$, one sees that $\tilde{C} \in K_{\mathfrak{d}}$ in both scenarios (1) and (2). Thus, by Lemma 4.3 again, $\tilde{C} \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, and there are finite $\tilde{\mathcal{F}}_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\tilde{C} \times Y \subseteq \bigcup_n \bigcup \tilde{\mathcal{F}}_n$. Thus,

$$\begin{aligned} C \times Y &\subseteq ((K \setminus H) \times Y) \cup ((C \cap H) \times Y) \\ &\subseteq ((K \setminus H) \times Y) \cup (\tilde{C} \times Y) \\ &\subseteq \bigcup_{n\in\mathbb{N}} \bigcup (\mathcal{F}_n \cup \tilde{\mathcal{F}}_n). \end{aligned}$$ □

Definition 4.5. Let κ be an infinite cardinal number. Let $\mathscr{K}_0(\kappa)$ be the family of regular spaces in K_κ. For successor ordinals $\alpha+1$, let $C \in \mathscr{K}_{\alpha+1}(\kappa)$ if C is regular, and:

(1) either there is a σ-compact $D \subseteq C$ with $C \setminus U \in \mathscr{K}_\alpha(\kappa)$ for all open $U \supseteq D$
(2) or C is a union of less than $\mathrm{cf}(\kappa)$ members of $\mathscr{K}_\alpha(\kappa)$.

For limit ordinals α, let $\mathscr{K}_\alpha(\kappa) = \bigcup_{\beta<\alpha} \mathscr{K}_\beta$.

For every α the class $\mathcal{K}_\alpha(\kappa)$ is closed under products with compact regular spaces. In particular, the classes $\mathscr{K}_\alpha(\kappa)$ are much wider than $\mathscr{C}_\alpha(\kappa)$. Babinkostova and Scheepers prove, essentially, that every member of $\mathscr{C}_{\aleph_0}(2)$ is in

$$\big(\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma), \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})\big)^\times$$

(see [1]).

Theorem 4.6. *The product of each member of* $\mathscr{K}_{\max\{\mathfrak{b},\mathfrak{g}\}}(\mathfrak{d})$ *with every member of* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ *satisfies* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.

Proof. We recall from the Scheepers Diagram that $\max\{\mathfrak{b},\mathfrak{g}\} \leq \operatorname{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})) \leq \operatorname{cf}(\mathfrak{d})$. A combination of the arguments in the proofs of Theorems 4.4 and 3.3 show that

$$\mathscr{K}_{\operatorname{add}(\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}))}(\mathfrak{d}) \subseteq \big(\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma), \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})\big)^{\times}.$$

Since we have already presented three proofs using these methods, we leave the verification to the reader. □

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References

[1] Liljana Babinkostova and Marion Scheepers, *Products and selection principles*, Topology Proc. **31** (2007), no. 2, 431–443. MR2476622 (2010b:54015)

[2] Tomek Bartoszyński and Haim Judah, *Set theory,* On the structure of the real line. A K Peters Ltd., Wellesley, MA, 1995. MR1350295 (96k:03002)

[3] A. Besicovitch, *Concentrated and rarified sets of points*, Acta Mathematica **62** (1933), 289–300. MR1555386

[4] A. S. Besicovitch, *Relations between concentrated sets and sets possessing property C*, Proc. Cambridge Philos. Soc. **38** (1942), 20–23. MR0005888 (3,226c)

[5] Andreas Blass, *Combinatorial cardinal characteristics of the continuum*, Handbook of set theory. Vols. 1, 2, 3, Springer, Dordrecht, 2010, pp. 395–489, DOI 10.1007/978-1-4020-5764-9_7. MR2768685

[6] Salvador García-Ferreira and Angel Tamariz-Mascarúa, *Some generalizations of rapid ultrafilters in topology and Id-fan tightness*, Tsukuba J. Math. **19** (1995), no. 1, 173–185. MR1346760 (96g:54030)

[7] J. Gerlits and Zs. Nagy, *Some properties of $C(X)$. I*, Topology Appl. **14** (1982), no. 2, 151–161, DOI 10.1016/0166-8641(82)90065-7. MR667661 (84f:54021)

[8] Winfried Just, Arnold W. Miller, Marion Scheepers, and Paul J. Szeptycki, *The combinatorics of open covers. II*, Topology Appl. **73** (1996), no. 3, 241–266, DOI 10.1016/S0166-8641(96)00075-2. MR1419798 (98g:03115a)

[9] Ljubiša D. R. Kočinac and Marion Scheepers, *Combinatorics of open covers. VII. Groupability*, Fund. Math. **179** (2003), no. 2, 131–155, DOI 10.4064/fm179-2-2. MR2029229 (2005c:54020)

[10] Ljubiša D. R. Kočinac, *Selected results on selection principles*, Proceedings of the 3rd Seminar on Geometry and Topology, Azarb. Univ. Tarbiat Moallem, Tabriz, 2004, pp. 71–104. MR2090207 (2005g:54038)

[11] Tal Orenshtein and Boaz Tsaban, *Linear σ-additivity and some applications*, Trans. Amer. Math. Soc. **363** (2011), no. 7, 3621–3637, DOI 10.1090/S0002-9947-2011-05228-1. MR2775821 (2012e:54028)

[12] Masami Sakai, *Property C'' and function spaces*, Proc. Amer. Math. Soc. **104** (1988), no. 3, 917–919, DOI 10.2307/2046816. MR964873 (89m:54004)

[13] Nadav Samet, Marion Scheepers, and Boaz Tsaban, *Partition relations for Hurewicz-type selection hypotheses*, Topology Appl. **156** (2009), no. 3, 616–623, DOI 10.1016/j.topol.2008.08.013. MR2492309 (2010e:05314)

[14] Marion Scheepers, *Combinatorics of open covers. I. Ramsey theory*, Topology Appl. **69** (1996), no. 1, 31–62, DOI 10.1016/0166-8641(95)00067-4. MR1378387 (97h:90123)

[15] Marion Scheepers, *Selection principles and covering properties in topology*, Note Mat. **22** (2003/04), no. 2, 3–41. MR2112729 (2005h:54024)

[16] Marion Scheepers, *Gerlits and function spaces*, Studia Sci. Math. Hungar. **47** (2010), no. 4, 529–557, DOI 10.1556/SScMath.47.2010.4.1161. MR2883130 (2012i:54001)
[17] Boaz Tsaban, *Some new directions in infinite-combinatorial topology*, Set theory, Trends Math., Birkhäuser, Basel, 2006, pp. 225–255, DOI 10.1007/3-7643-7692-9_7. MR2267150 (2007f:03064)
[18] Boaz Tsaban, *Additivity numbers of covering properties*, Selection principles and covering properties in topology, Quad. Mat., vol. 18, Dept. Math., Seconda Univ. Napoli, Caserta, 2006, pp. 245–282. MR2395757 (2009e:03088)
[19] Boaz Tsaban, *Menger's and Hurewicz's problems: solutions from "the book" and refinements*, Set theory and its applications, Contemp. Math., vol. 533, Amer. Math. Soc., Providence, RI, 2011, pp. 211–226, DOI 10.1090/conm/533/10509. MR2777750 (2012f:03096)
[20] Boaz Tsaban and Lyubomyr Zdomskyy, *Scales, fields, and a problem of Hurewicz*, J. Eur. Math. Soc. (JEMS) **10** (2008), no. 3, 837–866, DOI 10.4171/JEMS/132. MR2421163 (2009j:54037)

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