Topology and its Applications 165 (2014) 12–25

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Diagonalizations of dense families

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ARTICLE INFO

Article history:
Received 15 July 2013
Accepted 7 January 2014

MSC:
primary 37F20
secondary 26A03, 03E75, 03E17

Keywords:
Dense families
Selection principles
$\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$
Hurewicz property
Menger property
$\mathsf{S}_1(\mathrm{O},\mathrm{O})$
C''
Rothberger property
$\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$
$\mathsf{S}_{\mathrm{fin}}(\mathfrak{D},\mathfrak{D})$
$\mathsf{S}_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$
M-separable
Selectively separable
SS
$\mathsf{S}_1(\mathrm{D},\mathrm{D})$
$\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$
$\mathsf{S}_1(\mathcal{D},\mathcal{D})$
R-separable
$\mathsf{S}_{\mathrm{fin}}(\mathrm{D}_{\mathrm{o}},\mathrm{D})$
Tiny sequence
$\mathsf{S}_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$
$\mathsf{S}_1(\mathrm{D}_{\mathrm{o}},\mathrm{D})$
$\mathsf{S}_1(\mathcal{D},\mathcal{D})$
1-tiny sequence
Selectively c.c.c.
$\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{D})$
Weakly Hurewicz
Weakly Menger

ABSTRACT

We develop a unified framework for the study of classic and new properties involving diagonalizations of dense families in topological spaces. We provide complete classification of these properties. Our classification draws upon a large number of methods and constructions scattered in the literature, and on several novel results concerning the classic properties.

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0166-8641/$ – see front matter
http://dx.doi.org/10.1016/j.topol.2014.01.001

$\mathsf{S}_1(\mathrm{O},\mathrm{D})$
Weakly C″
$\mathsf{S}_1(\mathcal{O},\mathcal{D})$
Weakly Rothberger

1. Introduction

The following diagonalization prototypes are ubiquitous in the mathematical literature (see, e.g., the surveys [29,19,31]):

$\mathsf{S}_1(\mathscr{A},\mathscr{B})$: For all $\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathscr{A}$, there are $U_1\in\mathcal{U}_1, U_2\in\mathcal{U}_2,\ldots$ such that $\{U_n\colon n\in\mathbb{N}\}\in\mathscr{B}$.
$\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{B})$: For all $\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathscr{A}$, there are finite $\mathcal{F}_1\subseteq\mathcal{U}_1, \mathcal{F}_2\subseteq\mathcal{U}_2,\ldots$ such that $\bigcup_n\mathcal{F}_n\in\mathscr{B}$.

The papers [25,18] have initiated the simultaneous consideration of these properties in the case where $\mathscr{A}$ and $\mathscr{B}$ are important families of open covers of a topological space X. This unified study of topological properties, that were previously studied separately, had tremendous success, some of which were surveyed in the above-mentioned surveys. The field of *selection principles* is growing rapidly, and dozens of new papers appeared since these survey articles were published. The purpose of the present paper is to initiate a similar program for the case where $\mathscr{A}$ and $\mathscr{B}$ are dense families, as we now define.

Definition 1.1. Let X be a topological space. A family $\mathcal{U}\subseteq P(X)$ is a *dense family* if $\bigcup\mathcal{U}$ is a dense subset of X. A family $\mathcal{U}\subseteq P(X)$ is in:

D: if $\mathcal{U}$ is dense;
$\mathrm{D_o}$: if $\mathcal{U}$ is dense and all members of $\mathcal{U}$ are open; and
O: if $\mathcal{U}$ is an open cover of X.

In other words, $\mathcal{U}$ is a dense family if each open set in X intersects some member of $\mathcal{U}$. Note that

$$\mathrm{O}\subseteq\mathrm{D_o}\subseteq\mathrm{D}.$$

Every element of D is refined by a dense family of singletons. It follows, for example, that $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ is equivalent to the following property, studied under various names in the literature (see Table 1 below):

For each sequence A_n, $n\in\mathbb{N}$, such that $\overline{A_n}=X$ for all n, there are finite sets $F_n\subseteq A_n$, $n\in\mathbb{N}$, such that $\overline{\bigcup_n F_n}=X$.

We study all properties $\mathsf{S}(\mathscr{A},\mathscr{B})$ for $\mathsf{S}\in\{\mathsf{S}_1,\mathsf{S}_{\mathrm{fin}}\}$ and $\mathscr{A},\mathscr{B}\in\{\mathrm{O},\mathrm{D_o},\mathrm{D}\}$, by making use of their inter-connections. This approach is expected to have impact beyond these properties, not only concerning properties that imply or are implied by the above-mentioned properties (e.g., the corresponding game-theoretic properties), but also concerning formally unrelated properties that have a similar flavor.

The properties we are studying here were studied in the literature under various, sometimes pairwise incompatible, names. Examples are given in Table 1 below, with some references. We do not give references for $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ and $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, because there are hundreds of them. Instead, we refer to the above-mentioned surveys. In this table, by *obsolete* we mean that nowadays the name stands for another property.

A topological space is *$\mathscr{A}$-Lindelöf* ($\mathscr{A}\in\{\mathrm{D},\mathrm{D_o},\mathrm{O},\ldots\}$) if each member of $\mathscr{A}$ contains a countable member of $\mathscr{A}$. If X satisfies $\mathsf{S}_{\mathrm{fin}}(\mathscr{A},\mathscr{A})$, then X is $\mathscr{A}$-Lindelöf. Thus, $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ spaces are Lindelöf, $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ spaces are separable, and $\mathsf{S}_{\mathrm{fin}}(\mathrm{D_o},\mathrm{D})$ spaces are $\mathrm{D_o}$-Lindelöf, or equivalently, c.c.c. (i.e., such that every

Table 1
Earlier names of the studied properties.

Property	Classic names and references
$\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$	Hurewicz (obsolete), Menger
$\mathsf{S}_1(\mathrm{O},\mathrm{O})$	C'', Rothberger
$\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$	$\mathsf{S}_{\mathrm{fin}}(\mathfrak{D},\mathfrak{D})$ [27], $\mathsf{S}_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$ [3], M-separable [8,9,22] selectively separable (SS) [11,3,4,16,5,13,17,23,10,6]
$\mathsf{S}_1(\mathrm{D},\mathrm{D})$	$\mathsf{S}_1(\mathfrak{D},\mathfrak{D})$ [27], $\mathsf{S}_1(\mathcal{D},\mathcal{D})$ [3], R-separable [8,9,16,13,23]
$\mathsf{S}_{\mathrm{fin}}(\mathrm{D}_{\mathrm{o}},\mathrm{D})$	no tiny sequence [30,20], $\mathsf{S}_{\mathrm{fin}}(\mathcal{D},\mathcal{D})$ [26,28,23]
$\mathsf{S}_1(\mathrm{D}_{\mathrm{o}},\mathrm{D})$	$\mathsf{S}_1(\mathcal{D},\mathcal{D})$ [26,28,23], no 1-tiny sequence [20], selectively c.c.c. [2]
$\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{D})$	weakly Hurewicz (obsolete) [14,23], weakly Menger [21,24]
$\mathsf{S}_1(\mathrm{O},\mathrm{D})$	weakly C'' [14,23], $\mathsf{S}_1(\mathcal{O},\mathcal{D})$ [26], weakly Rothberger [21]

maximal disjoint family of open sets in the space is countable). For the latter assertion, note that every maximal disjoint family of open sets is dense, and every maximal open refinement of an element of D_{o} is a maximal disjoint family of open sets. This also explains why Aurichi's notion of *selectively c.c.c.* [2] is equivalent to $\mathsf{S}_1(\mathrm{D}_{\mathrm{o}},\mathrm{D})$.[2]

2. Classification

2.1. Implications

We begin with the 18 properties of the form $\mathsf{S}(\mathscr{A},\mathscr{B})$, where $\mathsf{S}\in\{\mathsf{S}_1,\mathsf{S}_{\mathrm{fin}}\}$ and $\mathscr{A},\mathscr{B}\in\{\mathrm{O},\mathrm{D}_{\mathrm{o}},\mathrm{D}\}$. We first observe that six of these properties are void, and consequently need not be considered.

Lemma 2.1. *Let X be a nondiscrete Hausdorff space. Then X does not satisfy any of the properties* $\mathsf{S}(\mathrm{D}_{\mathrm{o}},\mathrm{O})$, $\mathsf{S}(\mathrm{D},\mathrm{O})$*,* $\mathsf{S}(\mathrm{D},\mathrm{D}_{\mathrm{o}})$ $(\mathsf{S}\in\{\mathsf{S}_1,\mathsf{S}_{\mathrm{fin}}\})$.

Proof. Note that each of these properties $\mathsf{S}(\mathscr{A},\mathscr{B})$ implies that each member of $\mathscr{A}$ contains a countable member of $\mathscr{B}$. Let x be a nonisolated point.

Since $\{X\setminus\{x\}\}\in\mathrm{D}_{\mathrm{o}}\setminus\mathrm{O}$, X does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D}_{\mathrm{o}},\mathrm{O})$.

Let $y\in X\setminus\{x\}$, and let U,V be disjoint open neighborhoods of x,y respectively. Then the set $U\setminus\{x\}$ is not closed and not dense. Then $\mathcal{U}:=\{U\setminus\{x\},(U\setminus\{x\})^{\mathsf{c}}\}\in\mathrm{D}$, and each family of open sets contained in $\mathcal{U}$ contains at most $U\setminus\{x\}$, which is not dense. Thus, X does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D}_{\mathrm{o}})$.

The other assertions follow. □

The following immediate equivalences $(\mathsf{S}\in\{\mathsf{S}_1,\mathsf{S}_{\mathrm{fin}}\})$ eliminate the need to consider 4 additional properties:

$$\mathsf{S}(\mathrm{D}_{\mathrm{o}},\mathrm{D})=\mathsf{S}(\mathrm{D}_{\mathrm{o}},\mathrm{D}_{\mathrm{o}}),$$

$$\mathsf{S}(\mathrm{O},\mathrm{D})=\mathsf{S}(\mathrm{O},\mathrm{D}_{\mathrm{o}}).$$

We are thus left with the following eight properties.

[2] We thank Angelo Bella for bringing Aurichi's notion of selectively c.c.c. to our attention, and for pointing out its equivalence to $\mathsf{S}_1(\mathrm{D}_{\mathrm{o}},\mathrm{D})$.

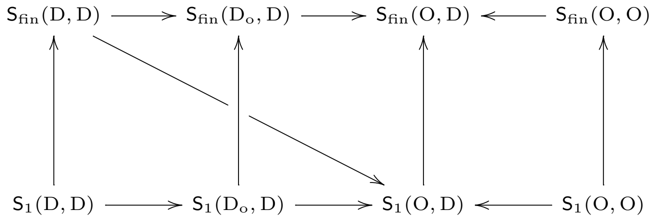


Fig. 1. The Dense Families Diagram.

For the diagonal implication, note that $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ implies separability, which in turn implies $\mathsf{S}_1(\mathrm{O},\mathrm{D})$. Indeed, every countable space satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, and we have the following.

Proposition 2.2. *Let $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$. If X has a dense subset satisfying $\mathsf{S}(\mathrm{O},\mathrm{O})$, then X satisfies $\mathsf{S}(\mathrm{O},\mathrm{D})$.* □

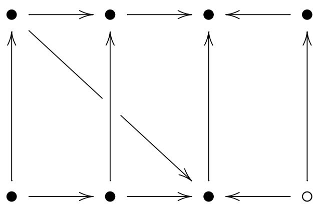
2.2. Non-implications

To make it clear which properties are possessed by the examples given below and which not, we supply a version of the Dense Families Diagram (Fig. 1) with a full bullet (•) for each property the example satisfies, and an empty bullet (◦) for each property not satisfied by the example.

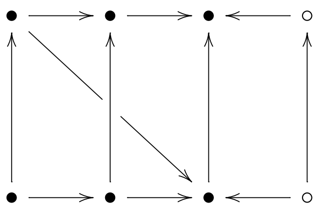
2.2.1. Uncountable examples

Proposition 2.3.

(1) *The spaces $\mathbb{R}$ and $\beta\mathbb{N}$ satisfy the following setting.*



(2) *The Baire space $\mathbb{N}^{\mathbb{N}}$ satisfies the following setting.*

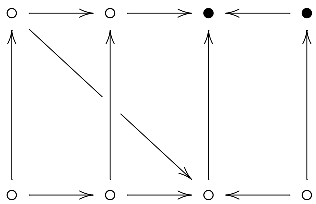


Proof. Each of these spaces has a countable pseudobase, and thus satisfies $\mathsf{S}_1(\mathrm{D},\mathrm{D})$ [11].

Being σ-compact, $\mathbb{R}$ and $\beta\mathbb{N}$ satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. Since $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ subsets of $\mathbb{R}$ have measure zero, $\mathbb{R}$ does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. For the same reason, the unit interval $[0,1]$ does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. Since $[0,1]$ (being separable and compact) is a continuous image of $\beta\mathbb{N}$ and $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ is preserved by continuous images, $\beta\mathbb{N}$ does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, too.

The Baire space does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ (e.g., [18]). □

Theorem 2.4. $\beta\mathbb{N}\setminus\mathbb{N}$ *satisfies the following setting.*



Proof. Being compact, $\beta\mathbb{N}\setminus\mathbb{N}$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. As $\beta\mathbb{N}\setminus\mathbb{N}$ is not c.c.c. (equivalently, not $\mathrm{D_o}$-Lindelöf) [15, Example 3.6.18], it does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D_o},\mathrm{D})$.

Lemma 2.5. $\beta\mathbb{N}\setminus\mathbb{N}$ *does not satisfy* $\mathsf{S}_1(\mathrm{O},\mathrm{D})$.

Proof. Let $[\mathbb{N}]^\infty$ be the family of all infinite subsets of $\mathbb{N}$. For $A\in[\mathbb{N}]^\infty$, let

$$[A]=\{p\in\beta\mathbb{N}\setminus\mathbb{N}\colon\ A\in p\}$$

be the standard basic clopen subset of $\beta\mathbb{N}\setminus\mathbb{N}$.

By induction on n, choose for each sequence $(s_1,\dots,s_n)\in\{0,1\}^n$ an infinite set $I_{s_1,\dots,s_n}$ such that:

(1) $I_0\cup I_1=\mathbb{N}$, and the union is disjoint.
(2) For each n and each sequence $(s_1,\dots,s_n)\in\{0,1\}^n$,

$$I_{s_1,\dots,s_n,0}\cup I_{s_1,\dots,s_n,1}=I_{s_1,\dots,s_n},$$

and this union is disjoint.

For each n, let

$$\mathcal{U}_n=\bigl\{[I_{s_1,\dots,s_n}]\colon\ s_1,\dots,s_n\in\{0,1\}\bigr\}.$$

$\mathcal{U}_n\in\mathrm{O}$. Indeed, since the involved unions are finite,

$$\bigcup_{(s_1,\dots,s_n)\in\{0,1\}^n}[I_{s_1,\dots,s_n}]=\Biggl[\bigcup_{(s_1,\dots,s_n)\in\{0,1\}^n}I_{s_1,\dots,s_n}\Biggr]=[\mathbb{N}]=\beta\mathbb{N}\setminus\mathbb{N}.$$

Now, consider any selection $[I_{s_1^n,\dots,s_n^n}]\in\mathcal{U}_n$, $n\in\mathbb{N}$. By induction on n, choose $t_n\in\{0,1\}$ such that

$$I_{t_1,\dots,t_n}\cap\bigl(I_{s_1^1}\cup I_{s_1^2,s_2^2}\cup\dots\cup I_{s_1^n,\dots,s_n^n}\bigr)=\emptyset$$

for all n. This is possible, since the latter union is contained in a union of at most $2^{n-1}+2^{n-2}+\dots+2+1=2^n-1$ sets of the form $I_{s_1,\dots,s_n}$. The sets $I_{t_1,\dots,t_n}$, $n\in\mathbb{N}$, form a decreasing sequence of infinite subsets of $\mathbb{N}$. Let A be a pseudointersection of these sets. For each n,

$$A\subseteq^* I_{t_1,\dots,t_n}\subseteq\bigl(I_{s_1^1}\cup I_{s_1^2,s_2^2}\cup\dots\cup I_{s_1^n,\dots,s_n^n}\bigr)^{\mathsf{c}}\subseteq I^{\mathsf{c}}_{s_1^n,\dots,s_n^n},$$

and thus $A\cap I_{s_1^n,\dots,s_n^n}$ is finite. Therefore, $[A]\cap[I_{s_1^n,\dots,s_n^n}]=[A\cap I_{s_1^n,\dots,s_n^n}]=\emptyset$ for all n. □

This completes the proof of Theorem 2.4. □

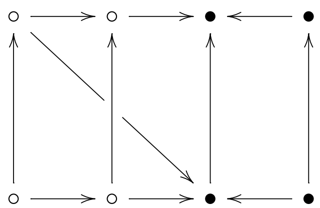
The properties $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ and $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ are hereditary for closed subsets (e.g., [18]). $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{D})$ is hereditary for compact subsets since compact spaces satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$. In contrast to that, we have the following.

Corollary 2.6. *None of the properties* $\mathsf{S}_1(\mathrm{D},\mathrm{D})$, $\mathsf{S}_1(\mathrm{D_o},\mathrm{D})$, $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$, $\mathsf{S}_{\mathrm{fin}}(\mathrm{D_o},\mathrm{D})$, $\mathsf{S}_1(\mathrm{O},\mathrm{D})$, *is hereditary for compact subsets.*

Proof. Proposition 2.3 and Theorem 2.4. □

We consider ordinals α with the order topology, so that the basic clopen sets are the intervals (β,γ) or $[0,\beta)$ or (β,α), where $\beta,\gamma\in\alpha$.

Theorem 2.7. *Each uncountable successor ordinal* $\alpha+1$ *satisfies the following setting.*



Proof. The theorem follows from the following two lemmata.

Lemma 2.8 ([12]). *Each uncountable successor ordinal* $\alpha+1$ *satisfies* $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.

Proof. For completeness, we reproduce the proof: Given open covers $\mathcal{U}_1,\mathcal{U}_2,\dots$ of $\alpha+1$, each consisting of basic clopen sets, pick $U_1\in\mathcal{U}_1$ with $\alpha\in U_1$. If $\alpha\setminus U_1\neq\emptyset$, it is a successor ordinal, and we can cover its last element by some $U_2\in\mathcal{U}_2$. This must end after finitely many steps, since the sequence of last elements is decreasing. □

Lemma 2.9 (Folklore). *For each uncountable ordinal* α, $\alpha+1$ *is not c.c.c. (equivalently, not* $\mathrm{D_o}$*-Lindelöf), having an uncountable dense set of isolated points.*

This completes the proof of Theorem 2.7. □

Consider the following construction from [26]. The *Alexandroff double* of $[0,1]$ is the space $[0,1]\times\{0,1\}$, with the basic open sets $\{(x,1)\}$ for each $x\in[0,1]$, and $(U\times\{0,1\})\setminus(F\times\{1\})$ for each open U in $[0,1]$ and each finite $F\subseteq[0,1]$. For each dense $X\subseteq[0,1]$, the subspace

$$T(X):=\bigl([0,1]\times\{0\}\bigr)\cup\bigl(X\times\{1\}\bigr)$$

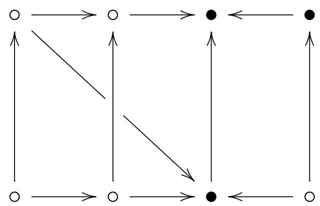
is compact Hausdorff.[3]

Theorem 2.10. *For a dense* $X\subseteq[0,1]$, $T(X)$ *satisfies the following*:

(1) *The setting in Proposition* 2.3(1) *if* X *is countable*;
(2) *The setting in Theorem* 2.4 *if* X *is uncountable and does not have strong measure zero*; *and*

[3] The notation $T(X)$ for this construction is due to Scheepers, in recognition of the inspiration provided by a related construction of Tkachuk.

(3) *The following setting if X is uncountable and has strong measure zero (e.g., when X is a Luzin set).*



Proof. The unit interval $[0,1]$ is a closed subspace of $T(X)$. Since $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ is preserved by moving to closed subsets, $T(X)$ does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. On the other hand, $T(X)$ is compact (after covering its lower part by finitely many sets, there remain only finitely many uncovered points on its top part), and thus satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.

(1) In this case, $T(X)$ has a countable base, and thus satisfies $\mathsf{S}_1(\mathrm{D},\mathrm{D})$.

(2, 3) If X is uncountable, then $T(X)$ is not c.c.c. (equivalently, not $\mathrm{D_o}$-Lindelöf), since $X \times \{1\}$ is an uncountable discrete dense subspace of $T(X)$. It remains to consider $\mathsf{S}_1(\mathrm{O},\mathrm{D})$, and this was done by Scheepers, who proved in [26] that $T(X)$ has this property if and only if X has strong measure zero. □

Daniels [14] proved that, for each $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$, if every finite subproduct of a product space $\prod_{i\in I} X_i$ satisfies $\mathsf{S}(\mathrm{O},\mathrm{D})$, then so does the full product $\prod_{i\in I} X_i$. It is a classic fact that the same assertion holds for *c.c.c.* (equivalently, $\mathrm{D_o}$-Lindelöf) spaces. We prove that this is also the case for $\mathsf{S}(\mathrm{D_o},\mathrm{D})$ ($\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$). Note that this is *not* the case for the remaining properties: Consider the countably infinite power $\mathbb{N}^{\mathbb{N}}$ of $\mathbb{N}$ for $\mathsf{S}(\mathrm{O},\mathrm{O})$ and the (nonseparable) power $\mathbb{N}^{\aleph_1}$ for $\mathsf{S}(\mathrm{D},\mathrm{D})$.

Modulo Lemma 2.12 below, whose proof is similar to that of Theorem 2.27 in [11], the following Theorem 2.11 was independently proved by Leandro Aurichi [2].

Theorem 2.11. *Let $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$. Let X_i, $i \in I$, be spaces such that $\prod_{i\in F} X_i$ satisfies $\mathsf{S}(\mathrm{D_o},\mathrm{D})$ for all finite $F \subseteq I$. Then $\prod_{i\in I} X_i$ satisfies $\mathsf{S}(\mathrm{D_o},\mathrm{D})$.*

Proof. We prove the assertion for $\mathsf{S} = \mathsf{S}_1$. The proof in the other case is similar.

For an open set U in a product space $\prod_i X_i$, let $\operatorname{supp}(U)$ (the *support* of U) be the finite set of coordinates i where $\pi_i(U) \neq X_i$. Note that open sets U, V in a product space intersect if and only if their projections $\pi_F(U), \pi_F(V)$ intersect, for $F = \operatorname{supp}(U) \cap \operatorname{supp}(V)$.

Lemma 2.12. *Let X_n, $n \in \mathbb{N}$, be spaces such that $\prod_{n\le k} X_n$ satisfies $\mathsf{S}_1(\mathrm{D_o},\mathrm{D})$ for all k. Then $\prod_{n\in\mathbb{N}} X_n$ satisfies $\mathsf{S}_1(\mathrm{D_o},\mathrm{D})$.*

Proof. Let $X = \prod_{n\in\mathbb{N}} X_n$, and let $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathrm{D_o}(X)$. Decompose $\mathbb{N} = \bigcup_{k\in\mathbb{N}} A_k$, with each A_k infinite.

Fix $k \in \mathbb{N}$. Since $\prod_{n\le k} X_n$ satisfies $\mathsf{S}_1(\mathrm{D_o},\mathrm{D})$ and $\{\pi_{\{1,\ldots,k\}}(U)\colon U \in \mathcal{U}_n\}$ is in $\mathrm{D_o}(\prod_{n\le k} X_n)$ for all $n \in A_k$, there are $U_n \in \mathcal{U}_n$, $n \in A_k$, such that $\{\pi_{\{1,\ldots,k\}}(U_n)\colon n \in A_k\} \in \mathrm{D}(\prod_{n\le k} X_n)$.

We claim that $\{U_n\colon n \in \mathbb{N}\}$ is a dense family in X. Indeed, let U be an open subset of X. Let k be such that $\operatorname{supp}(U) \subseteq \{1,\ldots,k\}$. By our construction, there is $n \in A_k$ such that the projections $\pi_{\{1,\ldots,k\}}(U)$ and $\pi_{\{1,\ldots,k\}}(U_n)$ intersect. Since $\operatorname{supp}(U) \subseteq \{1,\ldots,k\}$, U intersects U_n. □

We now prove the general assertion. Let $X = \prod_{i\in I} X_i$, and let $\mathcal{U}_1, \mathcal{U}_2, \ldots \in \mathrm{D_o}(X)$. Decompose $\mathbb{N} = \bigcup_{k\in\mathbb{N}} A_k$, with each A_k infinite.

Let I_1 be any countable nonempty subset of I. By the lemma, $\prod_{i\in I_1} X_i$ satisfies $\mathsf{S}_1(\mathrm{D_o},\mathrm{D})$. Thus, there are $U_n \in \mathcal{U}_n$, $n \in A_1$, such that $\{\pi_{I_1}(U_n)\colon n \in A_1\} \in \mathrm{D}(\prod_{i\in I_1} X_i)$. Let

$$I_2 = I_1 \cup \bigcup_{n \in A_1} \operatorname{supp}(U_n),$$

and note that I_2 is countable. By the lemma, $\prod_{i \in I_2} X_i$ satisfies $\mathsf{S}_1(\mathrm{D_o}, \mathrm{D})$. Thus, there are $U_n \in \mathcal{U}_n$, $n \in A_2$, such that $\{\pi_{I_2}(U_n)\colon n \in A_2\} \in \mathrm{D}(\prod_{i \in I_2} X_i)$. Let

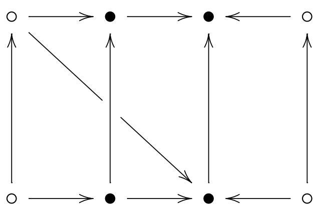
$$I_3 = I_2 \cup \bigcup_{n \in A_2} \operatorname{supp}(U_n).$$

Continue in the same manner.

We claim that $\{U_n\colon n \in \mathbb{N}\}$ is a dense family in X. Indeed, let U be an open subset of X. Let $I_\infty = \bigcup_{n \in \mathbb{N}} I_n$, and $F = \operatorname{supp}(U) \cap I_\infty$. Let k be such that $F \subseteq I_k$. By the construction, $\pi_F(U)$ intersects some $\pi_F(U_n)$, $n \in A_k$. Since $\operatorname{supp}(U_n) \subseteq I_\infty$, $\operatorname{supp}(U) \cap \operatorname{supp}(U_n) \subseteq F$. Thus, U intersects U_n. □

Theorem 2.13. *For each nonempty set X, the Tychonoff power $\mathbb{R}^X$ satisfies*:

(1) *The setting of Proposition 2.3*(1) *if X is finite nonempty*;
(2) *The setting of Proposition 2.3*(2) *if X is countably infinite*; *and*
(3) *The following setting if X is uncountable.*



Proof. If X is countable, then $\mathbb{R}^X$ has a countable base, and thus satisfies $\mathsf{S}_1(\mathrm{D}, \mathrm{D})$. If X is finite, then $\mathbb{R}^X$ is σ-compact, and thus satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Since $\mathbb{R}$ is a continuous image of $\mathbb{R}^X$, $\mathbb{R}^X$ does not satisfy $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. This concludes (1).

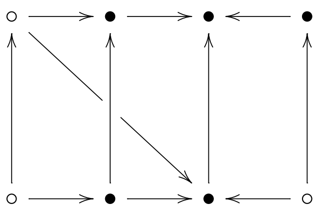
(2) $\mathbb{R}^{\mathbb{N}}$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

(3) As X is uncountable, $\mathbb{R}^X$ is not Lindelöf, and in particular does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. The Σ-product $\sum_{x \in X} \mathbb{R}$ (with respect to any point in $\mathbb{R}^X$) is a dense, nonseparable subset of $\mathbb{R}^X$. Thus, $\mathbb{R}^X$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D}, \mathrm{D})$.

It remains to prove that $\mathbb{R}^X$ satisfies $\mathsf{S}_1(\mathrm{D_o}, \mathrm{D})$, and this follows from Theorem 2.11.[4] □

Theorem 2.14. *The Tychonoff power $\{0,1\}^X$ satisfies*:

(1) *The setting of Proposition 2.3*(1) *if X is countably infinite*; *and*
(2) *The following setting if X is uncountable.*



[4] That $\mathbb{R}^X$ satisfies $\mathsf{S}_1(\mathrm{D_o}, \mathrm{D})$ was also, independently, proved by Aurichi [2].

Proof. $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ for $\{0,1\}^X$ follows from compactness.

The Cantor space does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, e.g., since $[0,1]$ is its continuous image. Thus, $\{0,1\}^X$ does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.

By Theorem 2.11, $\{0,1\}^X$ satisfies $\mathsf{S}_1(\mathrm{D}_\mathrm{o},\mathrm{D})$.

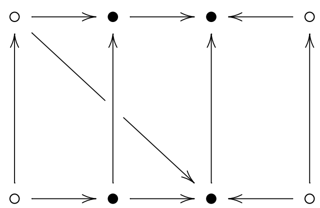
Finally, as in the previous proof, if X is uncountable, then the Σ-product $\sum_{x\in X}\{0,1\}$ (with respect to any point) is a dense, nonseparable subset of $\{0,1\}^X$. Thus, $\{0,1\}^X$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$.

$\{0,1\}^X$ is a compact space of uncountable π-weight, and thus does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ [11, Proposition 2.4]. □

For a topological space X, let Ω be the family of all $\mathcal{U}\in\mathrm{O}$ such that every finite subset of X is contained in some member of $\mathcal{U}$, and $X\notin\mathcal{U}$. Covering properties involving this family were studied extensively [29,19,31]. For a space X, $C_p(X)$ is the space of continuous real-valued functions on X, with the topology of pointwise convergence.

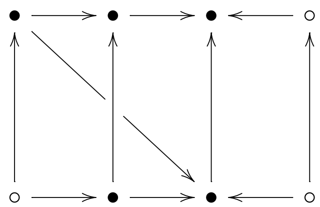
Theorem 2.15. *Let X be an infinite Tychonoff space. The space $C_p(X)$ satisfies:*

(1) *The setting*



if X does not satisfy $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ (e.g., if $X=\mathbb{N}^{\mathbb{N}}$) or there is no coarser, second countable Tychonoff topology on X;

(2) *The setting*



if X satisfies $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ but not $\mathsf{S}_1(\Omega,\Omega)$ and there is a coarser, second countable Tychonoff topology on X (e.g., if $X=\mathbb{R}$); and

(3) *The setting of Proposition 2.3(2) if X satisfies $\mathsf{S}_1(\Omega,\Omega)$ and there is a coarser, second countable Tychonoff topology on X.*

Proof. $C_p(X)$ is dense in $\mathbb{R}^X$. By Theorem 2.13, $C_p(X)$ satisfies $\mathsf{S}_1(\mathrm{D}_\mathrm{o},\mathrm{D})$. As X is infinite, $C_p(X)$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ [1, Theorem II.2.10].

By [8, Theorems 21,57], $C_p(X)$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ (respectively, $\mathsf{S}_1(\mathrm{D},\mathrm{D})$) if and only if there is a coarser, second countable Tychonoff topology on X and X satisfies $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ (respectively, $\mathsf{S}_1(\Omega,\Omega)$). □

Let $\mathbb{R}_{\mathrm{coc}}$ be $\mathbb{R}$ with the topology generated by the usual open intervals and all cocountable sets. This example was first considered in this context by Aurichi [2].

Proposition 2.16. *The space* $\mathbb{R}_{\mathrm{coc}}$ *satisfies the setting of Theorem* 2.14(2).

Proof. $\mathbb{R}_{\mathrm{coc}}$ is not separable, and thus does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$. Aurichi [2] proved that $\mathbb{R}_{\mathrm{coc}}$ satisfies $\mathsf{S}_1(\mathrm{D_o},\mathrm{D})$.

$\mathbb{R}_{\mathrm{coc}}$ does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ because $\mathbb{R}$, which is coarser, does not.

$\mathbb{R}_{\mathrm{coc}}$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ because $\mathbb{R}$ does: Given $\mathcal{U}_1,\mathcal{U}_2,\ldots\in\mathrm{O}(\mathbb{R}_{\mathrm{coc}})$, whose elements have the form $(a,b)\setminus C$ with C countable, let $\mathcal{U}'_1,\mathcal{U}'_2,\ldots$ be the open covers of $\mathbb{R}$ obtained by replacing each $(a,b)\setminus C$ with (a,b). Take finite $\mathcal{F}'_1\subseteq\mathcal{U}_1,\mathcal{F}'_2\subseteq\mathcal{U}'_2,\ldots$ such that $\bigcup_n\mathcal{F}'_n$ is a cover of $\mathbb{R}$. Then moving back to the original elements, we have that $\mathbb{R}\setminus\bigcup_n\mathcal{F}_n$ is countable. Choose one more element from each $\mathcal{U}_n$ to cover this countable remainder. □

Let X be a topological space. The *Pixley–Roy space* $\mathrm{PR}(X)$ is the space of all nonempty finite subsets of X, with the topology determined by the basic open sets

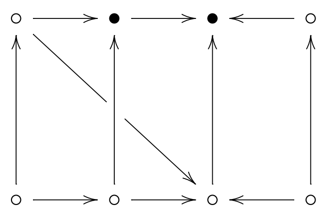
$$[F,U]:=\{H\in\mathrm{PR}(X):\ F\subseteq H\subseteq U\},$$

$F\in\mathrm{PR}(X)$ and U open in X.

For regular spaces X, the Pixley–Roy space $\mathrm{PR}(X)$ is zero-dimensional, completely regular, and hereditarily metacompact.

Theorem 2.17. *Let* X *be an uncountable separable metrizable space. The Pixley–Roy space* $\mathrm{PR}(X)$ *satisfies*:

(1) *The setting of Theorem* 2.15(1) *if* X *satisfies* $\mathsf{S}_1(\Omega,\Omega)$;
(2) *None of the properties if* X *does not satisfy* $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$; *and*
(3) *The following setting if* X *satisfies* $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ *but not* $\mathsf{S}_1(\Omega,\Omega)$ (*e.g., if* $X=\mathbb{R}$).



Proof. Daniels [14] proved that, for a metrizable space X, $\mathrm{PR}(X)$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{D})$ (respectively, $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{D})$) if and only if X satisfies $\mathsf{S}_1(\Omega,\Omega)$ (respectively, $\mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$).

Scheepers proved that, for Pixley–Roy spaces of separable metrizable spaces and $\mathsf{S}\in\{\mathsf{S}_1,\mathsf{S}_{\mathrm{fin}}\}$, $\mathsf{S}(\mathrm{D_o},\mathrm{D})=\mathsf{S}(\mathrm{O},\mathrm{D})$ [28].

If $\mathrm{PR}(X)$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$, then it is separable. It is a classic fact that, in this case, X is countable (references are available in [23]). Thus, in our case, $\mathrm{PR}(X)$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$.

Lemma 2.18. *The following are equivalent, for a topological space* X:

(1) $\mathrm{PR}(X)$ *has a countable cover by basic open sets*;
(2) $\mathrm{PR}(X)$ *is Lindelöf*;
(3) $\mathrm{PR}(X)$ *satisfies* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$;
(4) $\mathrm{PR}(X)$ *satisfies* $\mathsf{S}_1(\mathrm{O},\mathrm{O})$;
(5) $\mathrm{PR}(X)$ *is countable*; *and*
(6) X *is countable.*

Proof of (1) ⇒ (6). Assume that $\mathrm{PR}(X) = \bigcup_n [F_n, U_n]$. For each $x \in \mathrm{PR}(X)$, let n be such that $\{x\} \in [F_n, U_n]$. Then $F_n \subseteq \{x\}$, that is, $F_n = \{x\}$. It follows that there are only countably many singletons in $\mathrm{PR}(X)$, that is, X is countable. □

This completes the proof of Theorem 2.17. □

Remark 2.19. By recent results of Sakai [24], Theorem 2.17 generalizes from separable metrizable spaces to semi-stratifiable ones.

2.2.2. Countable examples

As pointed out already, Scheepers proved that, for Pixley–Roy spaces of separable metrizable spaces, $\mathsf{S}(\mathrm{D_o}, \mathrm{D}) = \mathsf{S}(\mathrm{O}, \mathrm{D})$ for both $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$ [28]. We prove an analogous assertion for countable spaces (note the difference in the properties involved).

The following theorem is, perhaps, the most surprising result in this paper.

Theorem 2.20. *Let $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$. Let X be a countable topological space. Then $\mathrm{PR}(X)$ satisfies $\mathsf{S}(\mathrm{D}, \mathrm{D})$ if and only if it satisfies $\mathsf{S}(\mathrm{D_o}, \mathrm{D})$.*

Proof. We prove the assertion for $\mathsf{S} = \mathsf{S}_1$. The proof of the remaining assertion is similar.

Assume that $\mathrm{PR}(X)$ satisfies $\mathsf{S}_1(\mathrm{D_o}, \mathrm{D})$, and let $D_1, D_2, \ldots$ be dense subsets of $\mathrm{PR}(X)$. Fix an enumeration $\mathrm{PR}(X) = \{H_n\colon n \in \mathbb{N}\}$, and a partition $\mathbb{N} = \bigcup_k I_k$ with each I_k infinite.

Fix k. For each $n \in I_k$, the family

$$\mathcal{U}_n = \{[F, X]\colon\ H_k \subseteq F \in D_n\}$$

is dense open in the subspaces $[H_k, X]$ of $\mathrm{PR}(X)$: For each basic open $[H, U]$ in $\mathrm{PR}(X)$ with $[H \cup H_k, U] = [H, U] \cap [H_k, X] \neq \emptyset$, let $F \in D_n \cap [H \cup H_k, U]$. Then $[F, X] \in \mathcal{U}_n$, and $F \cup H \cup H_k \in [F, X] \cap [H \cup H_k, U]$.

Since $\mathsf{S}_1(\mathrm{D_o}, \mathrm{D})$ is hereditary for open subsets, there are for each k elements $[F_n, X] \in \mathcal{U}_n$, $n \in I_k$, such that $\{[F_n, X]\colon\ n \in I_k\} \in \mathrm{D}([H_k, X])$. It remains to observe that $\{F_n\colon\ n \in \mathbb{N}\}$ is dense in $\mathrm{PR}(X)$. Indeed, let $[F, U]$ be a nonempty basic open set in $\mathrm{PR}(X)$. Let k be such that $H_k = F$. Since $[H_k, U]$ is open in $[H_k, X]$, there is $n \in I_k$ (so that $H_k \subseteq F_n$) such that

$$[F_n, U] = [F_n \cup H_k, U] = [F_n, X] \cap [H_k, U] \neq \emptyset.$$

Then $F_n \in [H_k, U]$. □

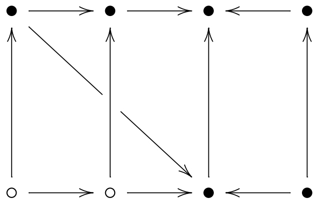
For a topological space X and a point $x \in X$, let $\pi\mathrm{N_{fin}}(x)$ be the family of all π-networks $\mathcal{N}$ of x (i.e., such that each open neighborhood of x contains an element of $\mathcal{N}$) such that all members of $\mathcal{N}$ are finite. For $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$, say that X satisfies $\mathsf{S}(\pi\mathrm{N_{fin}}, \pi\mathrm{N_{fin}})$ if $\mathsf{S}(\pi\mathrm{N_{fin}}(x), \pi\mathrm{N_{fin}}(x))$ holds for all $x \in X$.

Theorem 2.21 (Sakai [23])**.** *Let X be a countable topological space, and $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$. The following assertions are equivalent*:

(1) $\mathrm{PR}(X)$ *satisfies* $\mathsf{S}(\mathrm{D}, \mathrm{D})$;
(2) *X is countable, and all finite powers of X satisfy* $\mathsf{S}(\pi\mathrm{N_{fin}}, \pi\mathrm{N_{fin}})$.

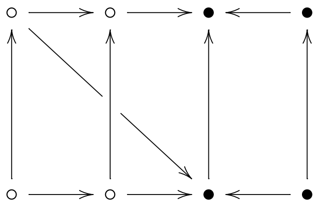
Theorem 2.22. *Let X be a countable topological space. The Pixley–Roy space* $\mathrm{PR}(X)$ *satisfies*:

(1) *All properties in the diagram if all finite powers of X satisfy* $\mathsf{S}_1(\pi\mathrm{N_{fin}}, \pi\mathrm{N_{fin}})$;
(2) *The setting*



if some finite power of X does not satisfy $\mathsf{S}_1(\pi\mathrm{N}_{\mathrm{fin}}, \pi\mathrm{N}_{\mathrm{fin}})$*, but all finite powers of X satisfy* $\mathsf{S}_{\mathrm{fin}}(\pi\mathrm{N}_{\mathrm{fin}}, \pi\mathrm{N}_{\mathrm{fin}})$;

(3) *The setting*



if some finite power of X does not satisfy $\mathsf{S}_{\mathrm{fin}}(\pi\mathrm{N}_{\mathrm{fin}}, \pi\mathrm{N}_{\mathrm{fin}})$.

Proof. Since X is countable, so is $\mathrm{PR}(X)$. Thus, $\mathrm{PR}(X)$ satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. Apply Theorems 2.20 and 2.21. □

To obtain concrete examples from Theorem 2.22, we use Nyikos' *Cantor Tree* topologies and a result of Sakai. Let $\{0,1\}^{<\infty}$ be the set of all finite sequences in $\{0,1\}$. For $s, t \in \{0,1\}^{<\infty}$, let $s \subseteq t$ mean that t is an end-extension of s.

Let $X \subseteq \{0,1\}^{\mathbb{N}}$, and define a topology on the countable space $\mathrm{CT}(X) := \{0,1\}^{<\infty} \cup \{\infty\}$ by declaring all points of $\{0,1\}^{<\infty}$ isolated, and taking the sets

$$\mathrm{CT}(X) \setminus \left(\{0,1\}^{\leq k} \cup \left\{s \in \{0,1\}^{<\infty}\colon\ \exists f \in F,\ s \subseteq f\right\}\right),$$

$k \in \mathbb{N}$, $F \subseteq X$ finite, as a local base at ∞.

Theorem 2.23 (Sakai [23])**.** *Let $X \subseteq \{0,1\}^{\mathbb{N}}$ and $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$. The following assertions are equivalent:*

(1) $\mathrm{CT}(X)$ *satisfies* $\mathsf{S}(\pi\mathrm{N}_{\mathrm{fin}}, \pi\mathrm{N}_{\mathrm{fin}})$;
(2) X *satisfies* $\mathsf{S}(\Omega, \Omega)$.

Corollary 2.24. *Let $X \subseteq \{0,1\}^{\mathbb{N}}$ and $\mathsf{S} \in \{\mathsf{S}_1, \mathsf{S}_{\mathrm{fin}}\}$. The following assertions are equivalent:*

(1) $\mathrm{PR}(\mathrm{CT}(X))$ *satisfies* $\mathsf{S}(\mathrm{D}_{\mathrm{o}}, \mathrm{D})$.
(2) $\mathrm{PR}(\mathrm{CT}(X))$ *satisfies* $\mathsf{S}(\mathrm{D}, \mathrm{D})$.
(3) X *satisfies* $\mathsf{S}(\Omega, \Omega)$.

Proof. The equivalence of (1) and (2) follows from Theorem 2.20. The equivalence of (2) and (3) follows from Sakai's Theorems 2.21 and 2.23. □

The first construction of a countable space not satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{D_o}, \mathrm{D})$ is due to Aurichi [2]. Our method makes it possible to transport examples from classic selection principles, and is consequently more flexible, as the following theorem shows.

Theorem 2.25. *Let* $X \subseteq \{0,1\}^{\mathbb{N}}$. *The countable space* $\mathrm{PR}(\mathrm{CT}(X))$ *satisfies*:

(1) *Setting* (2) *in Theorem 2.22 if* X *satisfies* $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ *but not* $\mathsf{S}_1(\Omega, \Omega)$ (*e.g.*, $X = \{0,1\}^{\mathbb{N}}$);
(2) *Setting* (3) *in Theorem 2.22 if* X *does not satisfy* $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$ (*e.g.*, $X = \mathbb{N}^{\mathbb{N}}$).

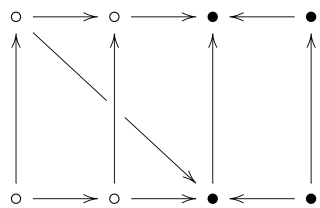
Proof. By Corollary 2.24. □

We conclude with an example of Barman and Dow [4]: Let $\mathbb{N} \cup \{\infty\}$ be the one-point compactification of $\mathbb{N}$. Take the box-product on $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$. Let

$$\mathbb{EI}^{\Box} = \{f \in (\mathbb{N} \cup \{\infty\})^{\mathbb{N}}\colon\ \exists n,\ f(1), \ldots, f(n) < \infty,\ f(n+1) = f(n+2) = \cdots = \infty\},$$

a countable subspace of the box-product space $(\mathbb{N} \cup \{\infty\})^{\mathbb{N}}$. $\mathbb{EI}^{\Box}$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D}, \mathrm{D})$ [4].

Theorem 2.26. *The Barman–Dow space* $\mathbb{EI}^{\Box}$ *satisfies the following setting.*



Proof. Since $\mathbb{EI}^{\Box}$ is countable, it satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. It Remains to prove that $\mathbb{EI}^{\Box}$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{D_o}, \mathrm{D})$. The proof is similar to the one above, due to Barman and Dow.

For n, let

$$U^n_m = \{f \in \mathbb{EI}^{\Box}\colon\ f(n) = m\}$$

for each $m \in \mathbb{N}$. Then $\mathcal{U}_n = \{U^n_m\colon\ m \in \mathbb{N}\}$ is an open dense family. Let $\mathcal{F}_n \subseteq \mathcal{U}_n$ be finite for all n. For each n, let m_n be maximal with $U^n_{m_n} \in \mathcal{F}_n$. Let

$$U = \mathbb{EI}^{\Box} \cap \prod_n [m_n + 1, \infty].$$

Then U is disjoint of $\bigcup \mathcal{F}_n$, for all n. □

Theorem 2.27. *No implication can be added to the Dense Families Diagram (Fig. 1), except for those obtained by composition of existing ones. Moreover, this is exhibited by ZFC examples.*

Proof. We go over the properties one by one, and verify that no new implication can be added from it to another property, by referring to the appropriate (one, in case there are several) proposition or theorem. When treating a property, we consider only potential implications not ruled out by the treatment of the previous properties.

(1) $\mathsf{S}_1(\mathrm{D},\mathrm{D}) \nrightarrow \mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$ (Proposition 2.3).
(2) $\mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D}) \nrightarrow \mathsf{S}_{\mathrm{fin}}(\mathrm{D_o},\mathrm{D})$ (Theorem 2.25).
(3) $\mathsf{S}_1(\mathrm{D_o},\mathrm{D}) \nrightarrow \mathsf{S}_{\mathrm{fin}}(\mathrm{D},\mathrm{D})$ (Theorem 2.13).
(4) $\mathsf{S}_{\mathrm{fin}}(\mathrm{D_o},\mathrm{D}) \nrightarrow \mathsf{S}_1(\mathrm{O},\mathrm{D})$ (Theorem 2.17).
(5) $\mathsf{S}_1(\mathrm{O},\mathrm{O}) \nrightarrow \mathsf{S}_{\mathrm{fin}}(\mathrm{D_o},\mathrm{D})$ (Theorem 2.7).
(6) $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O}) \nrightarrow \mathsf{S}_1(\mathrm{O},\mathrm{D})$ (Theorem 2.4). □

The classification is completed.

Acknowledgements

We thank Angelo Bella for useful discussions and for bringing Aurichi's results to our attention. We thank Leandro Aurichi for making his preprint [2] available to us. Proposition 2.16 is based on one by Aurichi. In addition to the above-cited papers, readers interested in the notions studies here may wish to also consult [7]. A part of this research was carried out during a visit of the third named author in the University of Messina. This author thanks his hosts, and in particular the other named authors and Andrei Catalioto, for their kind hospitality.

References

[1] A. Arhangel'skiĭ, Topological Function Spaces, Kluwer Academic Publishers, 1992.
[2] L. Aurichi, Selectively c.c.c. spaces, Topol. Appl. 160 (2013) 2243–2250.
[3] L. Babinkostova, On some questions about selective separability, Math. Log. Q. 55 (2009) 539–541.
[4] D. Barman, A. Dow, Selective separability and SS^+, Topol. Proc. 37 (2011) 181–204.
[5] D. Barman, A. Dow, Proper forcing axiom and selective separability, Topol. Appl. 159 (2012) 806–813.
[6] A. Bella, When is a Pixley–Roy hyperspace SS^+?, Topol. Appl. 160 (2013) 99–104.
[7] A. Bella, On two selection principles and the corresponding games, Topol. Appl. 160 (2013) 2309–2313.
[8] A. Bella, M. Bonanzinga, M. Matveev, Variations of selective separability, Topol. Appl. 156 (2009) 1241–1252.
[9] A. Bella, M. Bonanzinga, M. Matveev, Addendum to "Variations of selective separability" [Topology Appl. 156 (7) (2009) 1241–1252], Topol. Appl. 157 (2010) 2389–2391.
[10] A. Bella, M. Bonanzinga, M. Matveev, Sequential+separable vs. sequentially separable and another variation on selective separability, Cent. Eur. J. Math. 11 (2013) 530–538.
[11] A. Bella, M. Bonanzinga, M. Matveev, V. Tkachuk, Selective separability: general facts and behavior in countable spaces, Topol. Proc. 32 (2008) 15–30.
[12] M. Bonanzinga, F. Cammaroto, L. Kočinac, M. Matveev, On weaker forms of Menger, Rothberger and Hurewicz properties, Mat. Vesn. 61 (2009) 13–23.
[13] A. Bella, M. Matveev, S. Spadaro, Variations of selective separability II: Discrete sets and the influence of convergence and maximality, Topol. Appl. 159 (2012) 253–271.
[14] P. Daniels, Pixley–Roy spaces over subsets of the reals, Topol. Appl. 29 (1988) 93–106.
[15] R. Engelking, General Topology, Sigma Ser. Pure Math., vol. 6, Heldermann-Verlag, Berlin, 1989.
[16] G. Gruenhage, M. Sakai, Selective separability and its variations, Topol. Appl. 58 (2011) 1352–1359.
[17] G. Iurato, On density and π-weight of $L^p(\beta\mathbb{N},\mathbb{R},\mu)$, Appl. Gen. Topol. 13 (2012) 33–38.
[18] W. Just, A. Miller, M. Scheepers, P. Szeptycki, The combinatorics of open covers II, Topol. Appl. 73 (1996) 241–266.
[19] L. Kočinac, Selected results on selection principles, in: Sh. Rezapour (Ed.), Proceedings of the 3rd Seminar on Geometry and Topology, Tabriz, Iran, July 15–17, 2004, pp. 71–104.
[20] A. Kucharski, Universally Kuratowski–Ulam space and open–open games, arXiv:1202.2056, 2012.
[21] B. Pansera, Weaker forms of the Menger property, Quaest. Math. 34 (2011) 1–9.
[22] D. Repovš, L. Zdomskyy, On M-separability of countable spaces and function spaces, Topol. Appl. 157 (2010) 2538–2541.
[23] M. Sakai, Selective separability of Pixley–Roy hyperspaces, Topol. Appl. 159 (2012) 1591–1598.
[24] M. Sakai, The weak Hurewicz property of Pixley–Roy hyperspaces, Topol. Appl. 160 (2013) 2531–2537.
[25] M. Scheepers, Combinatorics of open covers I: Ramsey theory, Topol. Appl. 69 (1996) 31–62.
[26] M. Scheepers, Combinatorics of open covers (IV): subspaces of the Alexandroff double of the unit interval, Topol. Appl. 83 (1998) 63–75.
[27] M. Scheepers, Combinatorics of open covers (VI): Selectors for sequences of dense sets, Quaest. Math. 22 (1999) 109–130.
[28] M. Scheepers, Combinatorics of open covers (V): Pixley–Roy spaces and sets of reals, and omega-covers, Topol. Appl. 102 (2000) 13–31.
[29] M. Scheepers, Selection principles and covering properties in topology, Note Mat. 22 (2003) 3–41.
[30] A. Szymański, Some applications of tiny sequences, in: Proceedings of the 11th Winter School on Abstract Analysis, Rend. Circ. Mat. Palermo 2 (1984) 321–328.
[31] B. Tsaban, Some new directions in infinite-combinatorial topology, in: J. Bagaria, S. Todorčevic (Eds.), Set Theory, in: Trends Math., Birkhäuser, 2006, pp. 225–255.