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The linear refinement number and selection theory

by

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Abstract. The *linear refinement number* $\mathfrak{lr}$ is the minimal cardinality of a centered family in $[\omega]^\omega$ such that no linearly ordered set in $([\omega]^\omega, \subseteq^*)$ refines this family. The *linear excluded middle number* $\mathfrak{lx}$ is a variation of $\mathfrak{lr}$. We show that these numbers estimate the critical cardinalities of a number of selective covering properties. We compare these numbers to the classical combinatorial cardinal characteristics of the continuum. We prove that $\mathfrak{lr} = \mathfrak{lx} = \mathfrak{d}$ in all models where the continuum is at most $\aleph_2$, and that the cofinality of $\mathfrak{lr}$ is uncountable. Using the method of forcing, we show that $\mathfrak{lr}$ and $\mathfrak{lx}$ are not provably equal to $\mathfrak{d}$, and rule out several potential bounds on these numbers. Our results solve a number of open problems.

1. Overview

1.1. Combinatorial cardinal characteristics of the continuum. The definitions and basic properties not included below are available in [2].

A family $\mathcal{F} \subseteq [\omega]^\omega$ is *centered* if every finite subset of $\mathcal{F}$ has an infinite intersection. For $A, B \in [\omega]^\omega$, $B \subseteq^* A$ means that $B \setminus A$ is finite. A *pseudointersection* of a family $\mathcal{F} \subseteq [\omega]^\omega$ is an element $A \in [\omega]^\omega$ such that $A \subseteq^* B$ for all $B \in \mathcal{F}$. The *pseudointersection number* $\mathfrak{p}$ is the minimal cardinality of a centered family in $[\omega]^\omega$ that has no pseudointersection.

Definition 1.1 ([11, Definition 61]). A family $\mathcal{F} \subseteq [\omega]^\omega$ is *linear* if it is linearly ordered by $\subseteq^*$. A family $\mathcal{G} \subseteq [\omega]^\omega$ is a *refinement* of a family $\mathcal{F} \subseteq [\omega]^\omega$ if for each $A \in \mathcal{F}$ there is $B \in \mathcal{G}$ such that $B \subseteq^* A$. The *linear refinement number* $\mathfrak{lr}$ is the minimal cardinality of a centered family in $[\omega]^\omega$ that has no linear refinement.

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In [11], the ad-hoc name $\mathfrak{p}^*$ is used for the linear refinement number.

A *tower* is a linear subset of $[\omega]^\omega$ with no pseudointersection. The *tower number* $\mathfrak{t}$ is the minimal cardinality of a tower. It is immediate from the definitions that $\mathfrak{p} = \min\{\mathfrak{t}, \mathfrak{lr}\}$. Solving a longstanding problem, Malliaris and the second named author [5] have recently proved that $\mathfrak{p} = \mathfrak{t}$. We prove that, consistently, $\mathfrak{p} < \mathfrak{lr} < \mathfrak{c}$. This settles [11, Problem 64] (quoted in [10, Problem 5] and in [12, Problem 11.2 (311)]). Moreover, $\mathfrak{lr} = \mathfrak{d}$ in all models of set theory where the continuum is at most $\aleph_2$. One of our main results is that the cofinality of $\mathfrak{lr}$ is uncountable. The proof uses auxiliary results of independent interest. One striking consequence is that if $\mathfrak{p} < \mathfrak{b}$, then $\mathfrak{lr} \leq \mathfrak{b}$.

The number defined below is a variation of $\mathfrak{lr}$. It first appeared in [11, Problem 57], in the form non(wX).

DEFINITION 1.2. For functions $f, g \in \omega^\omega$, let $[f \leq g] = \{n : f(n) \leq g(n)\}$. The *linear excluded middle number* $\mathfrak{lx}$ is the minimal cardinality of a set of functions $\mathcal{F} \subseteq \omega^\omega$ such that, for each function $h \in \omega^\omega$, the family $\{[f \leq h] : f \in \mathcal{F}\}$ (is either not contained in $[\omega]^\omega$, or) does not have a linear refinement [1].

If $\mathcal{F} \subseteq \omega^\omega$ and $|\mathcal{F}| < \mathfrak{lx}$ then there are a function $h \in \omega^\omega$ and infinite subsets $A_f \subseteq^* [f \leq h]$ such that the family $\{A_f : f \in \mathcal{F}\}$ is linear, and for all functions $f, g \in \mathcal{F}$, say such that $A_f \subseteq^* A_g$, the function h excludes middles in the sense that

$$f(n) \leq h(n) < g(n)$$

may hold for at most finitely many n in A_f.

By the forthcoming Corollary 2.13, we have $\mathfrak{lr} \leq \mathfrak{lx} \leq \mathfrak{d}$ and $\mathfrak{b}, \mathfrak{s} \leq \mathfrak{lx}$.

In particular, the above-mentioned result on $\mathfrak{lr}$ implies that $\mathfrak{lx} = \mathfrak{d}$ whenever the continuum is at most $\aleph_2$. In light of the results of [6], Problem 57 in [11] asks whether $\mathfrak{lx} = \max\{\mathfrak{b}, \mathfrak{s}\}$. The answer, provided here, is “No”: In the model obtained by adding $\aleph_2$ Cohen reals to a model of the Continuum Hypothesis, $\mathfrak{b} = \mathfrak{s} = \aleph_1 < \mathfrak{d}$, and thus also $\mathfrak{b} = \mathfrak{s} < \mathfrak{lx} = \aleph_2$ in this model. This also answers the question whether $\mathsf{wX} = \mathsf{X}$, posed in [11] before Problem 57, since the critical cardinalities (defined below) of wX and of X are $\mathfrak{lx}$ and $\max\{\mathfrak{b}, \mathfrak{s}\}$, respectively.

For $\mathfrak{lx}$, an assertion finer than the above-mentioned one holds: If $\mathfrak{b} = \mathfrak{lx}$, then $\mathfrak{lx} = \mathfrak{d}$.

We use the method of forcing (necessarily, beyond continuum of size $\aleph_2$) to show that, consistently, $\mathfrak{lr}, \mathfrak{lx} < \mathfrak{d}$, and to rule out a number of potential

[1] In [11], the ad-hoc name *weak excluded middle number* ($\mathfrak{wx}$) is used for the linear excluded middle number. Since the excluded middle number defined in [11] turned out to be equal to the classical cardinal $\max\{\mathfrak{b}, \mathfrak{s}\}$, there is no point in preserving this name, and consequently also the name of its weaker version.

upper or lower bounds on these relatively new numbers in terms of classical combinatorial cardinal characteristics of the continuum. We conclude by stating a number of open problems.

1.2. Selective covering properties. Topological properties defined by diagonalizations of open or Borel covers have a rich history in various areas of general topology and analysis; see [9, 4, 12, 7] for surveys on the topic and some of its applications and open problems.

Let X be an infinite topological space. By a *cover* of X we mean a family $\mathcal{U}$ with $X \notin \mathcal{U}$ and $X = \bigcup \mathcal{U}$. Let $\mathcal{U} = \{U_n : n < \omega\}$ be a bijectively enumerated, countably infinite cover of X. We say that:

(1) $\mathcal{U} \in \mathrm{O}(X)$ if each U_n is open;
(2) $\mathcal{U} \in \Omega(X)$ if $\mathcal{U} \in \mathrm{O}(X)$ and each finite subset of X is contained in some U_n;
(3) $\mathcal{U} \in \mathrm{T}^*(X)$ if $\mathcal{U} \in \mathrm{O}(X)$, the sets

$$\{n : x \in U_n\} \quad (\text{for } x \in X)$$

are infinite, and the family of these sets has a linear refinement;
(4) $\mathcal{U} \in \Gamma(X)$ if $\mathcal{U}$ is a point-cofinite cover, that is, each element of X is a member of all but finitely many U_n.

We may omit the part "(X)" from these notations.

Let A and B be any of the above four types of open covers. Scheepers [8] introduced the following selection hypotheses that the space X may satisfy:

$\mathsf{S}_1(\mathrm{A}, \mathrm{B})$: For each sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of members of A, there is a selection $\langle U_n \in \mathcal{U}_n : n \in \mathcal{U}_n \rangle$ such that $\{U_n : n \in \omega\} \in \mathrm{B}$.

$\mathsf{S}_{\mathrm{fin}}(\mathrm{A}, \mathrm{B})$: For each sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of members of A, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n : n < \omega \rangle$ such that $\bigcup_{n<\omega} \mathcal{F}_n \in \mathrm{B}$.

$\mathsf{U}_{\mathrm{fin}}(\mathrm{A}, \mathrm{B})$: For each sequence $\langle \mathcal{U}_n : n < \omega \rangle$ of members of A which do not contain a finite subcover, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n : n < \omega \rangle$ such that $\{\bigcup \mathcal{F}_n : n \in \omega\} \in \mathrm{B}$.

Some of the properties are never satisfied, and many equivalences hold among the meaningful ones. The surviving properties appear in Figure 1, where an arrow denotes implication [11]. It is not known whether any implication, that does not follow from composition of existing ones, can be added to this diagram. Several striking results concerning this problem were established by Zdomskyy [13].

Below each property P in Figure 1 appears its *critical cardinality*, $\mathrm{non}(P)$, which is the minimal cardinality of a space X not satisfying that property ([2]).

([2]) The cardinal $\mathfrak{od}$ was defined in [6]. We recall the definition in Subsection 2.2, where it is needed.

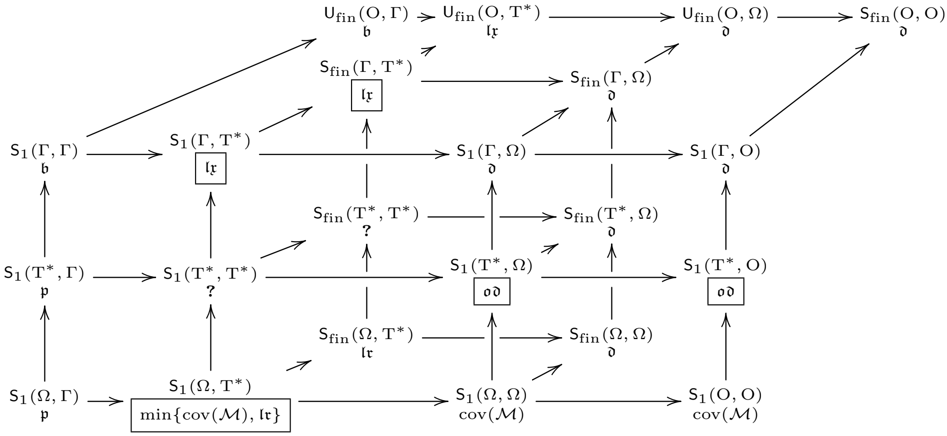

Fig. 1. The surviving properties

The boxed critical cardinalities, and several critical cardinalities of properties not displayed here, are established in the present paper.

Putting the above-mentioned results together, we find that in models where the continuum (or just $\mathfrak{d}$) is at most $\aleph_2$, all but one of the critical cardinalities of the properties under study are determined in terms of classical combinatorial cardinal characteristics of the continuum (see Figure 2).

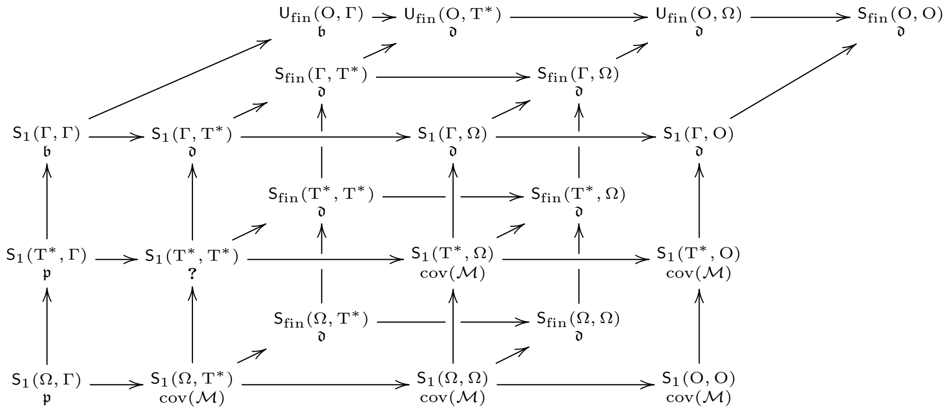

Fig. 2. The critical cardinalities in models of $\mathfrak{c} \leq \aleph_2$

These results fix, in particular, an erroneous assertion made in [11, Theorem 7.20] without proof, that the critical cardinality of $\mathsf{S}_1(\Omega,\mathrm{T}^*)$ is $\mathfrak{lt}$. As shown in the diagram, the correct critical cardinality is $\min\{\mathrm{cov}(\mathcal{M}),\mathfrak{lt}\}$. By the above-mentioned results, the inequality $\mathrm{cov}(\mathcal{M}) < \mathfrak{lt}$ holds in all models of $\mathrm{cov}(\mathcal{M}) < \mathfrak{d} = \aleph_2$, in particular in the standard Laver, Mathias, and Miller models (see [2]).

2. Results in ZFC

2.1. Combinatorial cardinal characteristics of the continuum. All filters in this paper are on ω, and *are assumed to contain all cofinite subsets of* ω. The *character* of a filter $\mathcal{F}$ is the minimal cardinality of a base for $\mathcal{F}$, that is, a set $\mathcal{B} \subseteq \mathcal{F}$ such that each element of $\mathcal{F}$ contains some element of $\mathcal{B}$, or equivalently, the minimal cardinality of a subset $\mathcal{B}$ of $\mathcal{F}$ generating $\mathcal{F}$ as a filter. Let $\mathcal{F}$ be a filter. A set $P \subseteq \omega$ is $\mathcal{F}$*-positive* if $P \cap A$ is infinite for all $A \in \mathcal{F}$; in other words, $\mathcal{F}$ can be extended to a filter containing P.

LEMMA 2.1. *Let κ be an infinite cardinal such that, for each filter $\mathcal{F}$ of character $\leq \kappa$, every linear subset of $\mathcal{F}$ of cardinality $< \kappa$ has an $\mathcal{F}$-positive pseudointersection. Then $\kappa < \mathfrak{lr}$.*

Proof. Let $\{A_\alpha : \alpha < \kappa\}$ be centered, and $\mathcal{F}$ be the filter generated by $\{A_\alpha : \alpha < \kappa\}$. We construct, by induction on α, a linear refinement $\{A_\alpha^- : \alpha < \kappa\}$ of $\{A_\alpha : \alpha < \kappa\}$ such that, for each α, $\{A_\beta^- : \beta < \alpha\} \cup \mathcal{F}$ is centered and $\{A_\beta^- : \beta < \alpha\}$ is a linear refinement of $\{A_\beta : \beta < \alpha\}$.

Let $A_0^- = A_0$. For $\alpha > 0$ we assume, inductively, that $\{A_\beta^- : \beta < \alpha\}$ is linear and that $\mathcal{F} \cup \{A_\beta^- : \beta < \alpha\}$ is centered. Let $\mathcal{F}_\alpha$ be the filter generated by $\mathcal{F} \cup \{A_\beta^- : \beta < \alpha\}$. Let P be an $\mathcal{F}_\alpha$-positive pseudointersection of $\{A_\beta^- : \beta < \alpha\}$. Take $A_\alpha^- = P \cap A_\alpha$. As $\mathcal{F}$ is a filter, A_α^- is $\mathcal{A}$-positive. ■

In the following proof, we use the fact that $\mathfrak{lr} \leq \mathfrak{d}$ [11]. Theorem 2.12 improves upon this inequality.

THEOREM 2.2. *If $\mathfrak{lr} = \aleph_1$, then $\mathfrak{d} = \aleph_1$.*

Proof. Assume $\mathfrak{d} > \aleph_1$. We will prove, using Lemma 2.1, that $\mathfrak{lr} > \aleph_1$. Let $\mathcal{F}$ be a filter of character $\leq \aleph_1$, and fix a base $\{B_\alpha : \alpha < \aleph_1\}$ of $\mathcal{F}$. Let $\{A_n : n < \omega\}$ be a linear subset of $\mathcal{F}$. By the previous lemma, it suffices to show that the family $\{A_n : n < \omega\}$ has an $\mathcal{F}$-positive pseudointersection. We may assume that $A_{n+1} \subseteq A_n$ for all n.

Let $\alpha < \aleph_1$. For each n, as $B_\alpha \cap A_n \in \mathcal{F}$, we can pick $f_\alpha(n) \in B_\alpha \cap A_n$ such that the function f_α is strictly increasing. As $\mathfrak{d} > \aleph_1$, there is a function $g \in \omega^\omega$ such that, for each $\alpha < \aleph_1$, $f_\alpha(n) < g(n)$ for infinitely many n. Let

$$P = \bigcup_{n<\omega} A_n \cap [0, g(n)).$$

For each n, $P \setminus A_n \subseteq \bigcup_{k<n} [0, g(k))$, and thus $P \subseteq^* A_n$. For each $\alpha < \aleph_1$ and each n with $f_\alpha(n) < g(n)$,

$$f_\alpha(n) \in B_\alpha \cap A_n \cap [0, g(n)) \subseteq P.$$

As f_α is strictly increasing, $B_\alpha \cap P$ is infinite. Thus, P is $\mathcal{F}$-positive. ■

As $\mathfrak{lr} \leq \mathfrak{d}$ (Corollary 2.13), Theorem 2.2 implies the following result.

Corollary 2.3. *If* $\mathfrak{d} \leq \aleph_2$*, then* $\mathfrak{lr} = \mathfrak{d}$. ∎

Thus, a large family of results about combinatorial cardinal characteristics of the continuum in models of $\mathfrak{c} = \aleph_2$ (see [2, Table 4]) are applicable. For example, we have the following consequences.

Corollary 2.4.

(1) *For each cardinal* $\mathfrak{x}$ *among* $\mathfrak{r}$, $\mathfrak{u}$, $\mathfrak{a}$, $\operatorname{cov}(\mathcal{N})$, $\operatorname{non}(\mathcal{N})$, *and* $\operatorname{non}(\mathcal{M})$, *it is consistent that* $\mathfrak{x} < \mathfrak{lr}$, *and that* $\mathfrak{lr} < \mathfrak{x}$.
(2) *For each cardinal* $\mathfrak{x}$ *among* $\mathfrak{p}$, $\mathfrak{h}$, $\mathfrak{s}$, $\mathfrak{g}$, $\mathfrak{e}$, $\mathfrak{b}$, $\operatorname{add}(\mathcal{N})$, $\operatorname{add}(\mathcal{M})$, *and* $\operatorname{cov}(\mathcal{M})$, *it is consistent that* $\mathfrak{x} < \mathfrak{lr}$.
(3) *For each cardinal* $\mathfrak{x}$ *among* $\mathfrak{i}$, $\operatorname{cof}(\mathcal{M})$, *and* $\operatorname{cof}(\mathcal{N})$, *it is consistent that* $\mathfrak{lr} < \mathfrak{x}$. ∎

In Subsection 3.1 we show that, consistently, $\mathfrak{lr} < \operatorname{cov}(\mathcal{M})$. In particular, $\mathfrak{lr} < \mathfrak{d}$ is consistent.

A *tower of height* κ is a set $\{T_\alpha : \alpha < \kappa\} \subseteq [\omega]^\omega$ that is $\subseteq^*$-decreasing with α and has no pseudointersection. There is no tower of height smaller than $\mathfrak{p}$, and by the Malliaris–Shelah Theorem, $\mathfrak{p}$ is the minimal height of a tower.

Lemma 2.5. *Let* $\mathcal{F} \subseteq [\omega]^\omega$ *be a centered family of cardinality smaller than* $\mathfrak{lr}$. *Then* $\mathcal{F}$ *is refined by a tower of height* $\mathfrak{p}$.

Proof. If $\mathfrak{p} = \mathfrak{lr}$, then $\mathcal{F}$ has a pseudointersection, and we can refine the pseudointersection by a tower of height $\mathfrak{t}$. In this case (or by the Malliaris–Shelah Theorem), since $\mathfrak{p} = \mathfrak{lr}$, we have $\mathfrak{p} = \mathfrak{t}$.

Assume that $\mathfrak{p} < \mathfrak{lr}$. Let $\{P_\alpha : \alpha < \mathfrak{p}\} \subseteq [\omega]^\omega$ be a centered family with no pseudointersection. Set

$$\mathcal{B} = \{A \times P_\alpha : A \in \mathcal{F},\ \alpha < \mathfrak{p}\} \cup \{\{(n,m) : k \leq \min\{n,m\}\} : k \in \omega\}.$$

Then $\mathcal{B}$ is a centered family of cardinality less than $\mathfrak{lr}$. Let $\mathcal{R} = \{R_\alpha : \alpha < \kappa\} \subseteq [\omega \times \omega]^\omega$ be a $\subseteq^*$-decreasing linear refinement of $\mathcal{B}$, with κ infinite and regular.

Let π_0 and π_1 be the projections of $\omega \times \omega$ on the first and second coordinates, respectively. For each pseudointersection R of the family $\{\{(n,m) : k \leq \min\{n,m\}\} : k \in \omega\}$, the sets $\pi_0(R)$ and $\pi_1(R)$ are both infinite. Moreover, if $R \subseteq^* A \times B$ then $\pi_0(R) \subseteq^* A$ and $\pi_1(R) \subseteq^* B$.

If $\kappa < \mathfrak{p}$, then $\mathcal{R}$ has a pseudointersection R. By the above paragraph, the set $A := \pi_1(R)$ is infinite, and is a pseudointersection of the family $\{P_\alpha : \alpha < \mathfrak{p}\}$; a contradiction.

Next, assume that $\mathfrak{p} \leq \kappa$. For each k, fix α_k such that $R_{\alpha_k} \subseteq^* \{(n,m) : k \leq \min\{n,m\}\}$. As κ is uncountable and regular, we see that $\alpha := \sup_k \alpha_k < \kappa$. Removing the first α members of $\mathcal{R}$, we may assume that every member of $\mathcal{R}$ is a pseudointersection of the family $\{\{(n,m) : k \leq$

$\min\{n,m\}\} : k \in \omega\}$, and consequently that the sets $\pi_0(R)$ and $\pi_1(R)$ are infinite for each $R \in \mathcal{R}$. It follows that the families $\{\pi_0(R_\alpha) : \alpha < \kappa\}$ and $\{\pi_1(R_\alpha) : \alpha < \kappa\}$ are linear refinements of $\mathcal{F}$ and $\{P_\alpha : \alpha < \mathfrak{p}\}$, respectively. In particular, if $\kappa = \mathfrak{p}$, then we are done.

It remains to prove that the case $\kappa > \mathfrak{p}$ is impossible. Assume otherwise. For each $\alpha < \mathfrak{p}$, fix $\beta_\alpha < \kappa$ such that $\pi_1(R_{\beta_\alpha}) \subseteq^* P_\alpha$. As κ is regular, we have $\beta := \sup_{\alpha<\mathfrak{t}} \beta_\alpha < \kappa$, and $\pi_1(R_\beta)$ is a pseudointersection of the family $\{P_\alpha : \alpha < \mathfrak{p}\}$; again a contradiction. ∎

LEMMA 2.6 (Folklore). *If $\mathfrak{b} < \mathfrak{d}$ then there is a tower of height $\mathfrak{b}$.*

Proof. Let $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$ be a $\mathfrak{b}$-*scale*, that is, an unbounded set where each f_α is an increasing function in ω^ω and the sequence f_α is $\le^*$-increasing with α. Let $h \in \omega^\omega$ witness that this family is not dominating. Then $\{[f_\alpha \le h] : \alpha < \mathfrak{b}\}$ (3) is a tower, for if P is a pseudointersection, then $\{f_\alpha\restriction P : \alpha < \mathfrak{b}\}$ is bounded by $h\restriction P$. ∎

THEOREM 2.7. *If $\mathfrak{p} < \mathfrak{b}$ then $\mathfrak{lr} \le \mathfrak{b}$.*

Proof. Assume that $\mathfrak{b} < \mathfrak{lr}$. Then, as $\mathfrak{lr} \le \mathfrak{d}$, we have $\mathfrak{b} < \mathfrak{d}$ and there is a tower $\{T_\alpha : \alpha < \mathfrak{b}\}$ of height $\mathfrak{b}$. By Lemma 2.5, this tower is refined by a tower $\{P_\alpha : \alpha < \mathfrak{p}\}$. Assume that $\mathfrak{p} < \mathfrak{b}$. For each $\alpha < \mathfrak{p}$, fix $\beta_\alpha < \mathfrak{b}$ with $P_\alpha \not\subseteq^* T_{\beta_\alpha}$. As $\mathfrak{b}$ is regular, $\beta := \sup_{\alpha<\mathfrak{p}} \beta_\alpha < \mathfrak{b}$. Then T_β is not refined by any P_α; a contradiction. ∎

The argument in the last proof shows the following.

COROLLARY 2.8. *Each tower of regular height smaller than $\mathfrak{lr}$ must be of height $\mathfrak{p}$.* ∎

A family of functions $\mathcal{F} \subseteq \omega^\omega$ is *κ-bounded* if there is a family $\mathcal{G} \subseteq \omega^\omega$ of cardinality κ such that each member of $\mathcal{F}$ is dominated by some member of $\mathcal{G}$.

LEMMA 2.9. *Let $\mathcal{F} \subseteq \omega^\omega$.*

(1) *If $|\mathcal{F}| < \mathfrak{lr}$, then $\mathcal{F}$ is $\mathfrak{p}$-bounded.*
(2) *If $\operatorname{cof}(\mathfrak{lr}) \le \mathfrak{p}$ and $|\mathcal{F}| = \mathfrak{lr}$, then $\mathcal{F}$ is $\mathfrak{p}$-bounded.*

Proof. (1) Let $\mathcal{F} \subseteq \omega^\omega$. We may assume that each member of $\mathcal{F}$ is an increasing function.

Assume that $|\mathcal{F}| < \mathfrak{lr}$. For each $f \in \mathcal{F}$, let

$$A_f = \{(n,m) : f(n) \le m\} \subseteq \omega \times \omega.$$

The family

$$\{A_f : f \in \mathcal{F}\} \cup \{\{(n,m) : n > k\} : k \in \omega\}$$

is centered.

(3) Recall that $[f \le h] = \{n \in \omega : f(n) \le h(n)\}$.

Assume that this family has a pseudointersection A. As A is a pseudointersection of $\{\{(n,m) : n > k\} : k \in \omega\}$, infinitely many columns $A \cap (\{n\} \times \omega)$ of A (for $n < \omega$) are nonempty, and all columns of A are finite. For each n, define $g_A(n)$ as follows: Let $n' \geq n$ be minimal with the column $A \cap (\{n'\} \times \omega)$ nonempty, and let $g_A(n)$ be minimal such that $(n', g_A(n))$ is in that column. For each $f \in \mathcal{F}$, as $A \subseteq^* A_f$ and f is increasing, we have $f \leq^* g_A$. Thus, $\mathcal{F}$ is bounded, and we are done.

Next, assume that our family does not have a pseudointersection. By Lemma 2.5, some tower $\{R_\alpha : \alpha < \mathfrak{p}\}$ linearly refines our family. As $\mathfrak{p}$ is regular, by removing an initial segment of indices we may assume that each R_α is a pseudointersection of $\{\{(n,m) : n > k\} : k \in \omega\}$. Thus, we can define functions g_{R_α} for $\alpha < \mathfrak{p}$ as in the previous paragraph. As above, for each $f \in \mathcal{F}$, if $\alpha < \mathfrak{p}$ is such that $R_\alpha \subseteq^* A_f$, then $f \leq^* g_{R_\alpha}$. This shows that $\mathcal{F}$ is $\mathfrak{p}$-bounded.

(2) Assume that $|\mathcal{F}| = \mathfrak{lr}$. Represent $\mathcal{F}$ as $\bigcup_{\alpha<\mathrm{cof}(\mathfrak{lr})} \mathcal{F}_\alpha$, with $|\mathcal{F}_\alpha| < \mathfrak{lr}$ for each α. Then every $\mathcal{F}_\alpha$ is $\mathfrak{p}$-bounded. As $\mathrm{cof}(\mathfrak{lr}) \cdot \mathfrak{p} = \mathfrak{p}$, $\mathcal{F}$ is $\mathfrak{p}$-bounded. ∎

THEOREM 2.10. *The cofinality of* $\mathfrak{lr}$ *is uncountable.*

Proof. As $\mathfrak{p}$ is regular, we know that $\mathfrak{lr}$ is regular if $\mathfrak{lr} = \mathfrak{p}$.

Assume that $\mathfrak{p} < \mathfrak{lr}$. Towards a contradiction, assume that $\mathrm{cof}(\mathfrak{lr}) = \aleph_0$. Let

$$\mathcal{F} = \{A_\alpha : \alpha < \mathfrak{lr}\} \subseteq [\omega]^\omega$$

be a centered family. We will prove that $\mathcal{F}$ has a linear refinement. Represent $\mathcal{F}$ as $\bigcup_n \mathcal{F}_n$ with $\mathcal{F}_n \subseteq \mathcal{F}_{n+1}$ and $|\mathcal{F}_n| < \mathfrak{lr}$ for all n. By thinning out the sequence $\langle \mathcal{F}_n : n < \omega \rangle$, we may assume that each $\mathcal{F}_n$ has a pseudointersection, or no $\mathcal{F}_n$ has a pseudointersection.

Consider first the former case. For each n, let R_n be a pseudointersection of $\mathcal{F}_n$. For each $A \in \mathcal{F}$, let k be the first with $A \in \mathcal{F}_k$. For $n < k$ let $f_A(n) = 0$, and for $n \geq k$ let

$$f_A(n) = \min\{m : R_n \setminus m \subseteq A\}.$$

By Lemma 2.9(2), the family $\{f_A : A \in \mathcal{F}\}$ is $\mathfrak{p}$-bounded. Let $\mathcal{G} \subseteq \omega^\omega$ be a witness for that. For each $g \in \mathcal{G}$ and each k, let

$$U_{g,k} = \bigcup_{n \geq k} R_n \setminus g(n).$$

The family $\{U_{g,k} : g \in \mathcal{G}, k \in \omega\}$ is centered. Indeed, for $k_1, \ldots, k_l \in \omega$ and $g_1, \ldots, g_l \in \omega^\omega$, let $n = \max\{k_1, \ldots, k_l\}$ and $m = \max\{g_1(n), \ldots, g_l(n)\}$. Then $n \geq k_1, \ldots, k_l$ and $R_n \setminus m \subseteq U_{g_1,k_1} \cap \cdots \cap U_{g_l,k_l}$. Since the cardinality of this family is at most $\mathfrak{p} < \mathfrak{lr}$, it has a linear refinement $\mathcal{R}$. Let $A \in \mathcal{F}$, and let $g \in \mathcal{G}$ be such that $f_A \leq^* g$. Fix k such that $f_A(n) \leq g(n)$ for all $n \geq k$. Then $U_{g,k} \subseteq A$. Thus, $\mathcal{R}$ is also a linear refinement of $\mathcal{F}$.

It remains to consider the case where no $\mathcal{F}_n$ has a pseudointersection. This is done by slightly extending the previous argument. By Lemma 2.5, for each n, there is a tower $\{T^n_\alpha : \alpha < \mathfrak{p}\}$ that linearly refines $\mathcal{F}_n$. Fix $A \in \mathcal{F}$, and let k be the first with $A \in \mathcal{F}_k$. For $n < k$ let $\alpha_n = 0$, and for $n \geq k$ let $\alpha_n < \mathfrak{p}$ be the first with $T^n_{\alpha_n} \subseteq^* A$. As $\mathfrak{p}$ is regular, the ordinal $\alpha(A) := \sup_n \alpha_n$ is smaller than $\mathfrak{p}$. Then

$$T^n_{\alpha(A)} \subseteq^* A$$

for all but finitely many n. For $n < k$ let $f_A(n) = 0$, and for $n \geq k$ let

$$f_A(n) = \min\{m : T^n_{\alpha(A)} \setminus m \subseteq A\}.$$

By Lemma 2.9, the family $\{f_A : A \in \mathcal{F}\}$ is $\mathfrak{p}$-bounded. Let $\mathcal{G} \subseteq \omega^\omega$ be a witness for that. For each $g \in \mathcal{G}$, $\alpha < \mathfrak{p}$ and $k \in \omega$, let

$$U_{g,\alpha,k} = \bigcup_{n \geq k} T^n_\alpha \setminus g(n).$$

The family $\{U_{g,\alpha,k} : g \in \mathcal{G},\ \alpha < \mathfrak{p},\ k \in \omega\}$ is centered, and has cardinality $\mathfrak{p} < \mathfrak{lr}$. Thus, it has a linear refinement $\mathcal{R}$. Let $A \in \mathcal{F}$, and let $g \in \mathcal{G}$ be such that $f_A \leq^* g$. Fix k such that $f_A(n) \leq g(n)$ for all $n \geq k$. Then $U_{g,\alpha(A),k} \subseteq A$. Thus, $\mathcal{R}$ is also a linear refinement of $\mathcal{F}$. ∎

We conclude this subsection with a result on $\mathfrak{lx}$ that is analogous to Theorem 2.2. Recall from Figure 1 that $\mathfrak{b} \leq \mathfrak{lx} \leq \mathfrak{d}$.

THEOREM 2.11. *If* $\mathfrak{lx} = \mathfrak{b}$ *then* $\mathfrak{d} = \mathfrak{b}$.

Proof. Assume that $\mathfrak{b} < \mathfrak{d}$. Let $\{f_\alpha : \alpha < \mathfrak{b}\} \subseteq \omega^\omega$. We will find a function $h \in \omega^\omega$ and a linear refinement of the family $\{[f_\alpha \leq h] : \alpha < \mathfrak{b}\}$.

For each $\alpha < \mathfrak{b}$, let g_α be a $\leq^*$-bound of $\{f_\alpha\} \cup \{g_\beta : \beta < \alpha\}$. Let $h \in \omega^\omega$ witness that $\{g_\alpha : \alpha < \mathfrak{b}\}$ is not dominating. Then $\{[g_\alpha \leq h] : \alpha < \mathfrak{b}\}$ is a linear refinement of $\{f_\alpha : \alpha < \mathfrak{b}\}$. ∎

As $\mathfrak{lr} \leq \mathfrak{lx} \leq \mathfrak{d}$, Corollary 2.4 holds for $\mathfrak{lx}$ as well. In Section 3.1 we show that, consistently, $\mathfrak{lx} < \mathfrak{d}$.

2.2. Selective covering properties. For a topological space X, let $\mathrm{T}(X)$ denote the family of all open covers $\{U_n : n < \omega\}$ of X such that the sets $\{n : x \in U_n\}$ (for $x \in X$) are infinite, and the family of these sets is linear. Recall that $\mathrm{T}^*(X)$ is the family of all open covers $\{U_n : n < \omega\}$ of X such that the sets $\{n : x \in U_n\}$ (for $x \in X$) are infinite, and the family of these sets has linear refinement. The first result of this section solves one of the first problems concerning this type of covers [11, Problem 10] (quoted in [12, Problem 7.2]).

For families of sets A and B, let $\binom{\mathrm{B}}{\mathrm{A}}$ denote the property that every element of B contains an element of A. The property $\binom{\mathrm{B}}{\mathrm{A}}$ becomes stronger if B is thinned out or A is extended.

THEOREM 2.12. *Let* $\mathrm{A} \subseteq \mathrm{T}^*$. *Then* $\binom{\Omega}{\mathrm{A}} = \mathsf{S}_{\mathrm{fin}}(\Omega, \mathrm{A})$. *In particular:*

(1) $\binom{\Omega}{\mathrm{T}} = \mathsf{S}_{\mathrm{fin}}(\Omega, \mathrm{T})$;
(2) $\binom{\Omega}{\mathrm{T}^*} = \mathsf{S}_{\mathrm{fin}}(\Omega, \mathrm{T}^*)$.

Proof. Clearly, $\mathsf{S}_{\mathrm{fin}}(\Omega, \mathrm{T}^*)$ implies $\binom{\Omega}{\mathrm{T}^*}$. It suffices to prove that $\binom{\Omega}{\mathrm{T}^*}$ implies $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$. Indeed, in this case $\binom{\Omega}{\mathrm{T}^*}$ implies

$$\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega) \cap \binom{\Omega}{\mathrm{T}^*} = \mathsf{S}_{\mathrm{fin}}(\Omega, \mathrm{T}^*).$$

Assume that $\{U^n_m : m \in \omega\} \in \Omega(X)$ for each $n < \omega$. Fix distinct elements $x_n \in X$ for $n < \omega$. Then

$$\mathcal{U} := \{U^n_m \setminus \{x_n\} : n, m \in \omega\} \in \Omega(X).$$

Let $\mathcal{V} \subseteq \mathcal{U}$ be such that $\mathcal{V} \in \mathrm{T}^*(X)$. Enumerate $\mathcal{V}$ as $\{V_n : n < \omega\}$. For $x \in X$, let $\mathcal{V}(x) = \{n : x \in V_n\}$. By the definition of T^*, the family $\{\mathcal{V}(x) : x \in X\}$ has a linear refinement $\mathcal{R}$.

There is a pseudointersection P of the family $\{\mathcal{V}(x_n) : n < \omega\}$ such that, for each finite $\mathcal{F} \subseteq \mathcal{R}$, $P \cap \bigcap \mathcal{F}$ is infinite. Indeed, if $\mathcal{R}$ has a pseudointersection then we can take P to be this pseudointersection. And if not, then by thinning $\mathcal{R}$ out, we may assume that $\mathcal{R} = \{R_\alpha : \alpha < \kappa\}$ is a tower of regular uncountable height κ. For each n, let $\alpha_n < \kappa$ satisfy $R_{\alpha_n} \subseteq^* \mathcal{V}(x_n)$. Let $\alpha = \sup_n \alpha_n$, and take $P = R_\alpha$.

Let $\mathcal{W} = \{V_k : k \in P\}$. Fix n. As $P \subseteq^* \mathcal{V}(x_n)$, we have $x_n \in V_k$ for all but finitely many $k \in P$. Thus, the set $\mathcal{W} \cap \{U^n_m \setminus \{x_n\} : m \in \omega\}$ is finite. Let $F_n \subseteq \omega$ be a finite (possibly empty) set such that

$$\mathcal{W} \cap \{U^n_m \setminus \{x_n\} : m \in \omega\} = \{U^n_m \setminus \{x_n\} : m \in F_n\}.$$

Let Y be a finite subset of X. Then the set $P \cap \bigcap_{y \in Y} \mathcal{V}(y)$ is infinite, and for each k in this set, $Y \subseteq V_k$. Thus, $\mathcal{W} \in \Omega(X)$. As $\mathcal{W} \in \Omega(X)$, the family $\bigcup_n \{U^n_m : m \in F_n\}$ is in $\Omega(X)$, too. ∎

COROLLARY 2.13.

(1) $\mathfrak{lr} \leq \mathfrak{lx} \leq \mathfrak{d}$;
(2) $\mathfrak{b}, \mathfrak{s} \leq \mathfrak{lx}$.

Proof. (1) First observe $\mathfrak{lr} = \mathrm{non}(\binom{\Omega}{\mathrm{T}^*})$, the critical cardinality of $\binom{\Omega}{\mathrm{T}^*}$. By Theorem 2.12, $\binom{\Omega}{\mathrm{T}^*}$ implies $\mathsf{U}_{\mathrm{fin}}(\Omega, \mathrm{T}^*)$, which is equivalent to $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathrm{T}^*)$. In [11, Theorem 55] it is proved, implicitly, that $\mathrm{non}(\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathrm{T}^*)) = \mathfrak{lx}$. This shows that $\mathfrak{lr} \leq \mathfrak{lx}$. Since $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathrm{T}^*)$ implies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, whose critical cardinality is $\mathfrak{d}$, we also have $\mathfrak{lx} \leq \mathfrak{d}$.

(2) By [11, Theorem 26] and [10, Theorem 9], we have $\mathrm{non}(\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathrm{T})) = \max\{\mathfrak{b}, \mathfrak{s}\}$. The property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathrm{T})$ implies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathrm{T}^*)$, whose critical cardinality is, as mentioned above, $\mathfrak{lx}$. ∎

THEOREM 2.14. *The critical cardinalities of* $\mathsf{S}_1(\Gamma, \mathrm{T}^*)$ *and* $\mathsf{S}_{\mathrm{fin}}(\Gamma, \mathrm{T}^*)$ *are both* $\mathfrak{l}\mathfrak{x}$.

Proof. As the critical cardinality of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathrm{T}^*)$ is $\mathfrak{l}\mathfrak{x}$ [11] and the implications

$$\mathsf{S}_1(\Gamma, \mathrm{T}^*) \to \mathsf{S}_{\mathrm{fin}}(\Gamma, \mathrm{T}^*) \to \mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathrm{T}^*)$$

hold, it suffices to prove that every topological space of cardinality smaller than $\mathfrak{l}\mathfrak{x}$ satisfies $\mathsf{S}_1(\Gamma, \mathrm{T}^*)$.

Let X be a topological space with $|X| < \mathfrak{l}\mathfrak{x}$. Assume that, for each n, $\{U^n_m : m < \omega\}$ is a point-cofinite cover of X. For each $x \in X$, define $f_x \in \omega^\omega$ by

$$f_x(n) = \min\{m : \forall k \geq m,\ x \in U^n_m\}.$$

As $|X| < \mathfrak{l}\mathfrak{x}$, there are $h \in \omega^\omega$ and infinite subsets

$$A_x \subseteq [f_x \leq h] \quad (x \in X)$$

such that $\{A_x : x \in X\}$ is linear. Then $\{U^n_{h(n)} : n < \omega\} \in \mathrm{T}^*(X)$. Indeed, for each $x \in X$,

$$A_x \subseteq [f_x \leq h] \subseteq \{n : x \in U^n_{h(n)}\},$$

and the family $\{A_x : x \in X\}$ is linear. ∎

THEOREM 2.15. *The critical cardinality of* $\mathsf{S}_1(\Omega, \mathrm{T}^*)$ *is* $\min\{\mathrm{cov}(\mathcal{M}), \mathfrak{l}\mathfrak{t}\}$.

Proof. Notice that

$$\mathsf{S}_1(\Omega, \mathrm{T}^*) = \mathsf{S}_1(\Omega, \Omega) \cap \binom{\Omega}{\mathrm{T}^*}.$$

It follows that

$$\mathrm{non}(\mathsf{S}_1(\Omega, \mathrm{T}^*)) = \min\bigl\{\mathrm{non}(\mathsf{S}_1(\Omega, \Omega)), \mathrm{non}\bigl(\binom{\Omega}{\mathrm{T}^*}\bigr)\bigr\}.$$

By the definitions of Ω and T^*, the critical cardinality of $\binom{\Omega}{\mathrm{T}^*}$ is $\mathfrak{l}\mathfrak{t}$ [11]. It is known that $\mathrm{non}(\mathsf{S}_1(\Omega, \Omega)) = \mathrm{cov}(\mathcal{M})$. ∎

THEOREM 2.16. $\min\{\mathrm{cov}(\mathcal{M}), \mathfrak{l}\mathfrak{t}\}, \min\{\mathfrak{b}, \mathfrak{s}\} \leq \mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*, \mathrm{T}^*)) \leq \mathfrak{l}\mathfrak{x}$.

Proof. As $\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*, \mathrm{T}^*)$ implies $\mathsf{S}_{\mathrm{fin}}(\Gamma, \mathrm{T}^*)$, we see that $\mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{T}, \mathrm{T}^*)) \leq \mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\Gamma^*, \mathrm{T}^*))$. By Theorem 2.14, $\mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\Gamma^*, \mathrm{T}^*)) \leq \mathfrak{l}\mathfrak{x}$. Thus, $\mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*, \mathrm{T}^*)) \leq \mathfrak{l}\mathfrak{x}$. By Theorem 2.15, as $\mathsf{S}_{\mathrm{fin}}(\Omega, \mathrm{T}^*)$ implies $\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*, \mathrm{T}^*)$, we deduce that $\min\{\mathrm{cov}(\mathcal{M}), \mathfrak{l}\mathfrak{t}\} \leq \mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*, \mathrm{T}^*))$. It remains to prove that $\min\{\mathfrak{b}, \mathfrak{s}\} \leq \mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*, \mathrm{T}^*))$. This is done as in [6, proof of Lemma 3.4]. For the reader's convenience, we provide a complete argument.

Let X be a topological space with $|X| < \min\{\mathfrak{b}, \mathfrak{s}\}$. Assume that, for each n, $\{U^n_m : m < \omega\} \in \mathrm{T}^*(X)$. For each n, let

$$A_x(n) \subseteq \{m : x \in U^n_m\} \quad (x \in X)$$

be a linear family. For $x, y \in X$, let

$$B_{x,y} = \{n : A_x(n) \subseteq^* A_y(n)\}.$$

As $|X| < \mathfrak{s}$, there is $S \in [\omega]^\omega$ that is not split by any $B_{x,y}$. As $B_{x,y} \cup B_{y,x} = \omega$, $S \subseteq^* B_{x,y}$ or $S \subseteq^* B_{y,x}$ for all x, y. For $x, y \in X$ define $g_{x,y} \in \omega^\omega$ by

$$g_{x,y}(n) = \begin{cases} \min\{k : A_x(n) \setminus k \subseteq A_y(n) \setminus k\}, & n \in B_{x,y} \setminus B_{y,x}, \\ \min\{k : A_y(n) \setminus k \subseteq A_x(n) \setminus k\}, & n \in B_{y,x} \setminus B_{x,y}, \\ \min\{k : A_x(n) \setminus k = A_y(n) \setminus k\}, & n \in B_{x,y} \cap B_{y,x}. \end{cases}$$

Since $|X| < \mathfrak{b}$, there exists $g_0 \in \omega^\omega$ which dominates all of the functions $g_{x,y}$ for $x, y \in X$. For each $x \in X$, define $g_x \in \omega^\omega$ by

$$g_x(n) = \min A_x(n) \setminus g_0(n).$$

Choose $g_1 \in \omega^\omega$ which dominates the functions g_x (for $x \in X$). Here too, this is possible since $|X| < \mathfrak{b}$. For each $n \in S$, let

$$\mathcal{F}_n = \{U^n_{g_0(n)}, \dots, U^n_{g_1(n)}\}.$$

For $n \notin S$ let $\mathcal{F}_n = \emptyset$. Let

$$\mathcal{U} = \bigcup_{n \in S} \mathcal{F}_n = \{U^n_m : n \in S,\ g_0(n) \le m \le g_1(n)\}.$$

We claim that $\mathcal{U} \in \mathrm{T}^*(X)$. For each $x \in X$ let

$$\mathcal{U}_x = \{U^n_m : n \in S,\ g_0(n) \le m \le g_1(n),\ m \in A_x(n)\} \subseteq \{U \in \mathcal{U} : x \in \mathcal{U}\}.$$

We may assume that the sets U^n_m are distinct for distinct pairs (n, m). For all but finitely many $n \in S$, $m := g_x(n) \in A_x(n)$ and $g_0(n) \le g_x(n) \le g_1(n)$, so $x \in U^n_m \in \mathcal{U}_x$. Thus, $\mathcal{U}_x$ is an infinite subset of $\mathcal{U}$. It remains to show that the family $\{\mathcal{U}_x : x \in X\}$ is linear.

Let $x, y \in X$. Without loss of generality, $S \subseteq^* B_{x,y}$. We will show that $\mathcal{U}_x \subseteq^* \mathcal{U}_y$. For all but finitely many $n \in S$, $g_{x,y}(n) \le g_0(n)$. For each $U^n_m \in \mathcal{U}_x$, $g_0(n) \le m \in A_x(n)$, and thus $g_{x,y}(n) \le m$. As $n \in B_{x,y}$ and $m \in A_x(n)$, we have $m \in A_y(n)$. Therefore, $U^n_m \in \mathcal{U}_y$. ∎

Definition 2.17 ([6]). The number $\mathfrak{od}$ is the minimal cardinality of a family $\mathcal{A} \subseteq ([\omega]^\omega)^\omega$ such that:

(1) for each n, $\{A(n) : A \in \mathcal{A}\}$ is linear,
(2) there is no $g \in \omega^\omega$ such that, for each $A \in \mathcal{A}$, $g(n) \in A(n)$ for some n.

Observe that $\mathrm{cov}(\mathcal{M}) \le \mathfrak{od}$, and equality holds if the continuum is at most $\aleph_2$ [6].

Theorem 2.18. *The critical cardinalities of* $\mathsf{S}_1(\mathrm{T}^*, \Omega)$ *and of* $\mathsf{S}_1(\mathrm{T}^*, \mathrm{O})$ *are both* $\mathfrak{od}$.

Proof. As $\mathrm{T} \subseteq \mathrm{T}^*$,

$$\mathsf{S}_1(\mathrm{T}^*, \Omega) \to \mathsf{S}_1(\mathrm{T}^*, \mathrm{O}) \to \mathsf{S}_1(\mathrm{T}, \mathrm{O}).$$

In [6] it is proved that $\mathrm{non}(\mathsf{S}_1(\mathrm{T}, \mathrm{O})) = \mathfrak{od}$. It remains to prove that $\mathfrak{od} \le \mathsf{S}_1(\mathrm{T}^*, \Omega)$.

Let X be a topological space with $|X| < \mathfrak{od}$. Assume that, for each n, $\{U^n_m : m < \omega\} \in \mathrm{T}^*(X)$. Fix n. By the definition of T^*, there are sets

$$A_x(n) \subseteq \{m : x \in U^n_m\}$$

such that $\{A_x(n) : x \in X\}$ is contained in $[\omega]^\omega$ and is linear. For each finite $F \subseteq X$, let

$$A_F(n) = \bigcap_{x \in F} A_x(n).$$

Then the family

$$\{A_F(n) : F \in [X]^{<\omega}\} \subseteq [\omega]^\omega$$

is linear. As $|X| < \mathfrak{od}$, there is $g \in \omega^\omega$ such that, for each finite $F \subseteq X$, there is n with $g(n) \in A_F(n)$. Then $\{U^n_{g(n)} : n < \omega\} \in \Omega(X)$. ∎

Recall that $\mathfrak{p} = \mathfrak{t}$ [5].

THEOREM 2.19. *The critical cardinality of* $\binom{\mathrm{T}^*}{\mathrm{T}}$ *is* $\mathfrak{t}$.

Proof. We use the method of [10, proof of Theorem 3].

($\geq$) $\binom{\mathrm{T}^*}{\Gamma}$, which implies $\binom{\mathrm{T}^*}{\mathrm{T}}$, has critical cardinality $\mathfrak{t}$.

($\leq$) Consider $P(\omega)$ with the Cantor space topology and the open sets

$$U_n = \{A \in P(\omega) : n \in A\}.$$

For a family $\mathcal{A} \subseteq [\omega]^\omega$, viewed as a subspace of $P(\omega)$:

- $\{U_n : n < \omega\} \in \mathrm{T}^*(\mathcal{A})$ if and only if $\mathcal{A}$ has a linear refinement;
- $\{U_n : n < \omega\} \in \mathrm{T}(\mathcal{A})$ if and only if $\mathcal{A}$ is linear;
- $\{U_n : n < \omega\}$ contains an element of $\mathrm{T}(\mathcal{A})$ if and only if there is $I \in [\omega]^\omega$ such that $\{A \cap I : A \in \mathcal{A}\}$ is a linear subset of $[\omega]^\omega$.

We construct a family $\mathcal{A} \subseteq [\omega]^\omega$ of cardinality $\mathfrak{t}$ such that $\mathcal{A}$ has a linear refinement, but for each $I \in [\omega]^\omega$ the family $\{A \cap I : A \in \mathcal{A}\}$ is nonlinear.

Let $\mathcal{F} \subseteq [\omega]^\omega$ be a tower of cardinality $\mathfrak{t}$. Let $\mathcal{B}$ be the boolean subalgebra of $P(\omega)$ generated by $\mathcal{F}$. Then $|\mathcal{B}| = \mathfrak{t}$. Let

$$\mathcal{A} = \{B \in \mathcal{B} : \exists A \in \mathcal{F},\, A \subseteq B\}.$$

Then $\mathcal{F}$ is a linear refinement of $\mathcal{A}$.

Towards a contradiction, assume that there is $I \in [\omega]^\omega$ such that $\{A \cap I : A \in \mathcal{A}\}$ is a linear subset of $[\omega]^\omega$. As $\{A \cap I : A \in \mathcal{A}\}$ refines $\mathcal{A}$, it has no pseudointersection. Fix $D_0 \in \mathcal{A}$. There exist:

- $D_1 \in \mathcal{A}$ such that $D_1 \cap I \subset^* D_0 \cap I$ (i.e., such that $D_0 \cap I \setminus D_1$ is infinite);
- $D_2 \in \mathcal{A}$ such that $D_2 \cap I \subset^* D_1 \cap I$.

Then the sets $(D_2 \cup (D_0 \setminus D_1)) \cap I$ and $D_1 \cap I$, both elements of $\mathcal{A}$, contain the infinite sets $(D_0 \cap I) \setminus (D_1 \cap I)$ and $(D_1 \cap I) \setminus (D_2 \cap I)$, respectively, and thus are not $\subseteq^*$-comparable; a contradiction. ∎

COROLLARY 2.20. *The critical cardinalities of* $\mathsf{S}_1(\mathrm{T}^*,\mathrm{T})$ *and* $\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*,\mathrm{T})$ *are both* $\mathfrak{t}$.

Proof. As

$$\mathsf{S}_1(\mathrm{T}^*,\mathrm{T}) = \binom{\mathrm{T}^*}{\mathrm{T}} \cap \mathsf{S}_1(\mathrm{T},\mathrm{T})$$

and $\mathrm{non}(\mathsf{S}_1(\mathrm{T},\mathrm{T})) = \mathfrak{t}$ [6], by Theorem 2.19 we have $\mathrm{non}(\mathsf{S}_1(\mathrm{T}^*,\mathrm{T})) = \mathfrak{t}$. Thus, by the implications

$$\mathsf{S}_1(\mathrm{T}^*,\mathrm{T}) \to \mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*,\mathrm{T}) \to \binom{\mathrm{T}^*}{\mathrm{T}}$$

and Theorem 2.19, $\mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*,\mathrm{T})) = \mathfrak{t}$. ∎

3. Consistency results

3.1. A model for $\mathfrak{lr} < \mathfrak{d}$. For a cardinal λ, let $\mathbb{C}_\lambda$ be the forcing notion adding λ Cohen reals.

THEOREM 3.1. *Let* $\mu = \mathfrak{c}$ *and* $\lambda > \mu^+$. *Then*

$$\Vdash_{\mathbb{C}_\lambda} \mathfrak{lr} \le \mu^+ < \lambda \le \mathrm{cov}(\mathcal{M}).$$

Proof. Let $\mathbb{C}_\lambda = \mathrm{Fn}(\lambda\times\omega,\omega)$ and let c_α be the αth Cohen real added by $\mathbb{C}_\lambda$. For $p \in \mathbb{C}_\lambda$, let $\mathrm{supp}(p) = \{\beta : \mathrm{dom}(p) \cap (\{\beta\}\times\omega) \ne \emptyset\}$. For $\beta \in \mathrm{supp}(p)$, let $p(\beta)$ be the partial function from ω to ω defined by $p(\beta)(n) = p(\beta,n)$. Thus, if $(\beta,n) \in \mathrm{dom}(p)$ and $p(\beta)(n) = m$, then $p \Vdash \dot{c}_\beta(n) = m$.

For the (standard) proof of the inequality $\lambda \le \mathrm{cov}(\mathcal{M})$, we refer to [2, p. 472].

We claim that the set $\{c_\alpha : \alpha < \mu^+\}$ witnesses that $\mathfrak{lr} \le \mu^+$. Towards a contradiction, assume that there are: A condition $p \in \mathbb{C}_\lambda$, a name $\dot{h}$ for a function in ω^ω, and names $\dot{A}_\alpha$ (for $\alpha < \mu^+$) of infinite subsets of ω such that:

(i) $p \Vdash \dot{A}_\alpha \subseteq^* \{n \in \omega : \dot{c}_\alpha(n) \le \dot{h}(n)\}$ and $\dot{A}_\alpha$ is infinite;
(ii) for all α and β, we have $p \Vdash \dot{A}_\alpha \subseteq \dot{A}_\beta$ or $\dot{A}_\beta \subseteq \dot{A}_\alpha$.

Fix $U_h \in [\lambda]^{\aleph_0}$ and a Borel function $b_h\colon (\omega^\omega)^{U_h} \to \omega^\omega$, coded in the ground model, such that

$$p \Vdash \dot{h} = b_h(\langle \dot{c}_\beta : \beta \in U_h\rangle).$$

For each $\alpha < \lambda$, fix a set $U_\alpha \in [\lambda]^{\aleph_0}$ containing U_h and a Borel function $b_\alpha\colon (\omega^\omega)^{U_\alpha} \to P(\omega)$, coded in the ground model, such that

$$p \Vdash \dot{A}_\alpha = b_\alpha(\langle \dot{c}_\beta : \beta \in U_\alpha\rangle).$$

Using the Δ-System Lemma, find $W \in [\mu^+]^{\mu^+}$ and U_* with $U_\alpha \cap U_\beta = U_*$ for all distinct $\alpha,\beta \in W$. As $U_h \subseteq U_\alpha$ for each α, we have $U_h \subseteq U_*$.

Fix distinct $\alpha, \beta \in W$ such that
$$\alpha \notin U_\beta \quad \text{and} \quad \beta \notin U_\alpha.$$
This can be done as follows: Select any $\alpha \in W \setminus U_*$ and distinct $\beta_0, \beta_1 \in W \setminus U_\alpha$. If $\alpha \notin U_{\beta_0}$, then set $\beta = \beta_0$. Otherwise, $\alpha \notin U_{\beta_1}$ because $U_{\beta_0} \cap U_{\beta_1} = U_*$. In this case, set $\beta = \beta_1$. We know that
$$p \Vdash \dot{A}_\alpha \subseteq^* \dot{A}_\beta \text{ or } \dot{A}_\beta \subseteq^* \dot{A}_\alpha.$$
There is $p_0 \leq p$ such that
$$(p_0 \Vdash \dot{A}_\alpha \subseteq^* \dot{A}_\beta) \quad \text{or} \quad (p_0 \Vdash \dot{A}_\beta \subseteq^* \dot{A}_\alpha).$$
Without loss of generality, we may assume that
$$p_0 \Vdash \dot{A}_\alpha \subseteq^* \dot{A}_\beta.$$
Take n_1 and a condition $p_1 \leq p_0$ such that:

- $p_1 \Vdash \dot{A}_\alpha \setminus n_1 \subseteq \dot{A}_\beta \setminus n_1$;
- $\alpha, \beta \in \operatorname{supp}(p_1)$.

Choose n_2 and a condition $p_2 \leq p_1$ such that:

- $n_2 > \max\{n_1, \max \operatorname{dom}(p_1(\beta))\}$;
- $p_2 \Vdash n_2 \in \dot{A}_\alpha$.

We know that $p \Vdash \dot{A}_\alpha = b_\alpha(\langle \dot{c}_\beta : \beta \in U_\alpha \rangle)$, and thus $\dot{A}_\alpha$ is a $(\mathbb{C}_\lambda)_{U_\alpha}$-name where
$$(\mathbb{C}_\lambda)_U := \{q \in \mathbb{C}_\lambda : \operatorname{supp}(q) \subseteq U\}.$$
Thus, we may assume that
$$p_2 \restriction \lambda \setminus U_\alpha = p_1 \restriction \lambda \setminus U_\alpha.$$
As $\dot{h}$ is a $(\mathbb{C}_\lambda)_{U_h}$-name, there are m_* and a condition $p_3 \leq p_2$ such that:

- $p_3 \restriction \lambda \setminus U_h = p_2 \restriction \lambda \setminus U_h$;
- $p_3 \Vdash \dot{h}(n_2) = m_*$.

Finally, choose $p_4 \in \mathbb{C}_\lambda$ such that:

- $\operatorname{supp}(p_4) = \operatorname{supp}(p_3)$;
- $p_4 \restriction \lambda \setminus \{\beta\} = p_3 \restriction \lambda \setminus \{\beta\}$;
- $p_4(\beta) = \sigma$, where $\sigma \colon \operatorname{dom}(p_3(\beta)) \cup \{n_2\} \to \omega$ is defined by
$$\sigma(k) = \begin{cases} p_3(\beta)(k), & k \in \operatorname{dom}(p_3(\beta)), \\ m_* + 1, & k = n_2. \end{cases}$$

In summary, the condition p_4 forces that:

(1) $\dot{A}_\alpha \setminus n_1 \subseteq \dot{A}_\beta \setminus n_1$;
(2) $n_2 \in \dot{A}_\alpha$;
(3) $\dot{A}_\beta \subseteq^* \{n \in \omega : \dot{c}_\beta(n) \leq \dot{h}(n)\}$;
(4) $\dot{h}(n_2) = m_*$;
(5) $\dot{c}_\beta(n_2) = m_* + 1$.

Conditions (4) and (5) imply that

$$p_4 \Vdash n_2 \notin \{n \in \omega : \dot{c}_\beta(n) \le \dot{h}(n)\}.$$

On the other hand $p \Vdash n_2 \in \dot{A}_\alpha$. Taking into account (1) and the fact that $n_2 > n_1$, we get $p \Vdash n_2 \in \dot{A}_\beta$. This, together with (3), implies that

$$p_4 \Vdash n_2 \in \{n \in \omega : \dot{c}_\beta(n) \le \dot{h}(n)\};$$

a contradiction. ∎

COROLLARY 3.2. *Let V be a model of the Continuum Hypothesis. For each cardinal $\lambda > \aleph_2$ of uncountable cofinality,*

$$V^{\mathbb{C}_\lambda} \models \mathfrak{p} = \mathfrak{b} = \aleph_1 < \mathfrak{lr} = \mathfrak{lx} = \aleph_2 < \lambda = \mathrm{cov}(\mathcal{M}) = \mathfrak{d} = \mathfrak{c}.$$

Proof. By Theorem 3.1, $\mathfrak{lr} \le \mathfrak{lx} \le \aleph_2$ in $V^{\mathbb{C}_\lambda}$. By Theorem 2.2, $\mathfrak{lr} > \aleph_1$ in $V^{\mathbb{C}_\lambda}$. The remaining assertions are well known (see, e.g., [2, §11]). ∎

3.2. A model for $\mathfrak{p} \ll \mathfrak{lr}$. Our model will be constructed using Mathias-type forcing notions. For a centered family $\mathcal{F}$ which contains all cofinite sets, the *$\mathcal{F}$-Mathias forcing* is the c.c.c. forcing notion

$$\mathbb{P} = \{\langle v, A\rangle \in [\omega]^{<\omega} \times \mathcal{F} : \max v < \min A\},$$

ordered by

$$\langle u, B\rangle \le \langle v, A\rangle \quad \text{if and only if} \quad u \supseteq v,\ B \subseteq A \text{ and } u \setminus v \subseteq A.$$

This forcing notion adds a pseudointersection to $\mathcal{F}$. Indeed, if G is $\mathbb{P}$-generic, then $\bigcup\{v : \langle v, A\rangle \in G\}$ is a pseudointersection of $\mathcal{F}$.

THEOREM 3.3. *Assume the Generalized Continuum Hypothesis, and let μ, κ and λ be uncountable cardinal numbers such that $\kappa = \mathrm{cof}(\kappa) < \mu = \mathrm{cof}(\mu) < \lambda = \lambda^{<\mu}$. There is a c.c.c. forcing notion $\mathbb{P}$ of cardinality λ such that*

$$\Vdash_{\mathbb{P}} \mathfrak{p} = \mathfrak{b} = \kappa < \mathfrak{lr} = \mu < \lambda = \mathfrak{c}.$$

Proof. Instead of building a model directly, as in the previous section, we will consider a transfinite sequence of classes of forcing notions, Θ_ξ, and with their help we will define the forcing notion we are looking for.

A forcing notion $\mathbb{O}$ belongs to the class Θ_ξ if $\mathbb{O}$ is given by an iteration $\mathcal{I}$ such that:

(1) $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \le \lambda\cdot\xi,\ \beta < \lambda\cdot\xi\rangle$ is a finite support iteration of length $\lambda\cdot\xi$ (ordinal product);
(2) $\mathbb{O} = \mathbb{P}_{\lambda\cdot\xi}$;
(3) $\mathbb{P}_0$ is the trivial forcing;
(4) for each $\alpha < \lambda\cdot\xi$, $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is an $\dot{\mathcal{F}}_\alpha$-Mathias forcing;
(5) $\dot{\mathcal{F}}_\alpha$ is a name for a filter generated by the cofinite sets together with the family $\{\dot{A}_{\alpha,\iota} : \iota < \iota_\alpha\}$, where ι_α is an ordinal $< \mu$;

(6) $\iota_\alpha = 0$ for $\alpha < \lambda$ (thus $\mathbb{Q}_\alpha$ is isomorphic to Cohen's forcing for $\alpha < \lambda$);
(7) $\dot{A}_{\alpha,\iota}$ is a $\mathbb{P}_\alpha$-name for a subset of ω;
(8) $b_{\alpha,\iota}\colon (2^\omega)^\omega \to [\omega]^\omega$ is a Borel function from the Cantor cube $(2^\omega)^\omega$ into $[\omega]^\omega$, coded in the ground model;
(9) $\Vdash_{\mathbb{P}_\alpha} \dot{A}_{\alpha,\iota} = b_{\alpha,\iota}(\langle \dot{B}_{\gamma(\alpha,\iota,n)} : n < \omega\rangle)$, where $B_\gamma \subseteq [\omega]^\omega$ denotes the γth generic real;
(10) if $\alpha = \lambda\cdot\zeta + \nu$ (where $\nu < \lambda$), then

$$\gamma(\alpha,\iota,n) < \lambda\cdot\zeta;$$

(11) for each $\zeta < \xi$ and each sequence $\langle b_\iota : \iota < \iota_*\rangle$ of Borel functions $b_\iota\colon (2^\omega)^\omega \to [\omega]^\omega$ of length $\iota_* < \mu$, and all ordinal numbers $\delta(\iota,n) < \lambda\cdot\zeta$ such that $\mathbb{P}$ forces that the filter generated by the cofinite sets together with the family

$$\{b_\iota(\langle \dot{B}_{\delta(\iota,n)} : n < \omega\rangle) : \iota < \iota_*\}$$

is proper, there are arbitrarily large $\alpha < \lambda\cdot(\zeta+1)$ such that:

- $\iota_\alpha = \iota_*$;
- $b_{\alpha,\iota} = b_\iota$ for all $\iota < \iota_*$;
- $\gamma(\alpha,\iota,n) = \delta(\iota,n)$ for all $\iota < \iota_*$ and all n.

If a forcing notion $\mathbb{O} \in \Theta_\xi$ is obtained by an iteration $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \le \lambda\cdot\xi,\ \beta < \lambda\cdot\xi\rangle$, then we set $\mathbb{O}_\alpha = \mathbb{Q}_\alpha$ for all α.

We say that a forcing $\mathbb{X}$ is the *restriction* of a forcing $\mathbb{O}$ to an ordinal ξ, $\mathbb{X} = \mathbb{O}{\restriction}\xi$, if there is $\zeta \ge \xi$ such that $\mathbb{X} \in \Theta_\xi$, $\mathbb{O} \in \Theta_\zeta$ and $\mathbb{O}_\alpha = \mathbb{X}_\alpha$ for all $\alpha < \lambda\cdot\xi$.

To complete the proof of the theorem, we prove several lemmata.

LEMMA 3.4. *The classes Θ_ξ have the following properties:*

(i) *if $\mathbb{O} \in \Theta_\xi$ and $\zeta < \xi$, then $\mathbb{O}{\restriction}\zeta \in \Theta_\zeta$;*
(ii) *Θ_0 is nonempty;*
(iii) *if $\mathbb{O} \in \Theta_\xi$, then there is $\mathbb{X} \in \Theta_{\xi+1}$ such that $\mathbb{X}{\restriction}\xi = \mathbb{O}$;*
(iv) *if ξ is a limit ordinal and $\langle \mathbb{O}^\zeta : \zeta < \xi\rangle$ is a sequence of forcing notions such that $\mathbb{O}^\zeta \in \Theta_\zeta$ and $\mathbb{O}^\zeta{\restriction}\eta = \mathbb{O}^\eta$ for all $\eta < \zeta < \xi$, then there is a unique $\mathbb{O}^\xi \in \Theta_\xi$ such that $\mathbb{O}^\xi{\restriction}\zeta = \mathbb{O}^\zeta$ for all $\zeta < \xi$.*

Proof. The only nontrivial property is (iii). To define $\mathbb{X}$, it suffices to find functions $b_{\alpha,\iota}$ and numbers $\gamma(\alpha,\iota,n)$ for $\alpha \in [\lambda\cdot\xi, \lambda\cdot(\xi+1))$ such that conditions (10) and (11) hold. Let $\langle \mathcal{P}_\alpha : \lambda\cdot\xi \le \alpha < \lambda\cdot(\xi+1)\rangle$ be the sequence of all possible pairs

$$\mathcal{P} = \big\langle \langle b_\iota : \iota < \iota_*\rangle, \langle \delta(\iota,n) : \iota < \iota_*, n < \omega\rangle \big\rangle$$

where

(a) $\iota_* < \mu$;
(b) $\langle b_\iota : \iota < \iota_* \rangle$ is a sequence of Borel functions;
(c) $\langle \delta(\iota, n) : \iota < \iota_*,\ n < \omega \rangle$ is a matrix of ordinal numbers $\delta(\iota, n) < \lambda \cdot \xi$;
(d) the filter generated by the cofinite sets and the family

$$\{b_\iota(\langle \dot{B}_{\delta(\iota,n)} : n < \omega \rangle) : \iota < \iota_*\}$$

is proper.

We request that each pair appears cofinally often in this sequence. When

$$\mathcal{P}_\alpha = \big\langle \langle b_\iota : \iota < \iota_* \rangle, \langle \delta(\iota, n) : \iota < \iota_*, n < \omega \rangle \big\rangle,$$

write $b_{\alpha,\iota} = b_\iota$, $\gamma(\alpha, \iota, n) = \delta(\iota, n)$ and $\iota_\alpha = \iota_*$. ∎

Using the above lemma, take a sequence $\langle \mathbb{O}^\xi : \xi \le \kappa \rangle$ of forcing notions such that $\mathbb{O}^\xi \in \Theta_\xi$ and $\mathbb{O}^\xi \restriction \zeta = \mathbb{O}^\zeta$ for every $\zeta < \xi$. Let $\mathbb{P}_\alpha = \mathbb{O}^\kappa_\alpha$ for all $\alpha \le \lambda \cdot \kappa$. The forcing notions $\mathbb{P}_\alpha$ are well defined: $\mathbb{O}^\xi_\alpha = \mathbb{O}^\kappa_\alpha$ for $\xi < \kappa$ and $\alpha < \lambda \cdot \xi$.

Lemma 3.5. $\Vdash_{\mathbb{P}_{\lambda\cdot\kappa}} \mathfrak{p} = \mathfrak{b} = \kappa$.

Proof. ($\mathfrak{p} \ge \kappa$) Let $\mathcal{A} = \{A_\iota : \iota < \iota_*\} \in V[G]$ be a centered family of cardinality $< \kappa$. Let $\xi_0 < \lambda \cdot \kappa$ be such that $\mathcal{A} \in V[G_{\xi_0}]$ (ξ_0 exists since we consider finite support iteration and $\kappa = \mathrm{cof}(\kappa) > \aleph_0$). We claim that there is $\alpha > \xi_0$ such that $\mathcal{A} \subseteq \mathcal{F}_\alpha$. Indeed, consider functions $b_\iota \colon (2^\omega)^\omega \to [\omega]^\omega$ and ordinals $\delta(\iota, n)$ such that

$$b_\iota(\langle B_{\delta(\iota,n)} : n < \omega \rangle) = A_\iota$$

for all $\iota < \iota_*$. By condition (11), there is α such that $\iota_\alpha = \iota_*$, $b_{\alpha,\iota} = b_\iota$ and $\delta(\iota, n) = \gamma(\alpha, \iota, n)$ for all $\iota < \iota_*$ and all n. Thus, B_α is a pseudointersection of $\mathcal{A}$.

($\mathfrak{b} \le \kappa$) Let $f_\xi \in \omega^\omega$ be an enumeration of $B_{\lambda\cdot\xi}$ in $V[G]$. Then the family $\{f_\xi : \xi < \kappa\}$ is unbounded in $V^{\mathbb{O}_{\lambda\cdot(\xi+1)}}$. The family $\langle \dot{c}_\xi : \xi < \kappa \rangle$ is forced to be unbounded. ∎

Lemma 3.6. $\Vdash_{\mathbb{P}_{\lambda\cdot\kappa}} \mathfrak{lr} \ge \mu$.

Proof. Assume that some $p \in \mathbb{P}_{\lambda\cdot\xi}$ forces that a family $\mathcal{A} = \{A_\iota : \iota < \iota_*\}$, where $\iota_* < \mu$, is contained in $[\omega]^\omega$ and closed under finite intersections. There are numbers $\delta(\iota, n) < \lambda \cdot \kappa$ (for $\iota < \iota_*$ and $n < \omega$) and Borel functions $b_\iota \colon (2^\omega)^\omega \to [\omega]^\omega$ such that

$$p \Vdash_{\mathbb{P}_{\lambda\cdot\kappa}} \dot{A}_\iota = b_\iota(\langle \dot{B}_{\delta(\iota,n)} : n < \omega \rangle) \text{ for all } \iota.$$

Write each $\delta(\iota, n)$ in the form

$$\delta(\iota, n) = \lambda \cdot \zeta(\iota, n) + \eta(\iota, n),$$

where $\eta(\iota, n) < \lambda$. Set

$$\eta_* = \sup\{\eta(\iota, n) : \iota < \iota_*,\ n < \omega\}.$$

As $\mathrm{cof}(\lambda) \geq \mu$ (indeed, $\lambda = \lambda^{<\mu}$), we have $\eta_* < \lambda$. Since, in addition, μ is regular, there is $S \subseteq \lambda \cdot \kappa$ of cardinality $< \mu$ such that

- $\{\delta(\iota, n) : \iota < \iota_*,\, n < \omega\} \subseteq S$;
- if $\alpha \in S$ then $\{\gamma(\alpha, \iota, n) : \iota < \iota_\alpha,\, n < \omega\} \subseteq S$.

Set

$$S_\xi = S \cap \lambda \cdot \xi.$$

Then $\langle S_\xi : \xi < \kappa\rangle$ is a $\subseteq$-increasing sequence. Let $U_\xi = \{\iota < \iota_* : \forall n,\, \delta(\iota, n) \in S_\xi\}$, so that $\langle U_\xi : \xi < \kappa\rangle$ is $\subseteq$-increasing with union ι_*.

Choose by induction β_ξ and η_ξ, $\xi < \kappa$, such that

- $\beta_\xi = \lambda \cdot \xi + \eta_\xi$ where $\eta_\xi < \lambda$,
- $\beta_\xi \notin S$,
- $\eta_\xi > \sup(\{\eta_\zeta : \zeta < \xi\} \cup \{\eta_*\})$, and
- $\Vdash_{\mathbb{P}_{\lambda\cdot\xi}} \{\dot{A}_{\beta_\xi,\iota} : \iota < \iota_{\beta_\xi}\} = \{\dot{A}_\iota : \iota \in U_\xi\} \cup \{\dot{B}_{\beta_\zeta} : \zeta < \xi\}$.

The induction can be carried out since

$$\Vdash_{\mathbb{P}_{\lambda\cdot\kappa}} \dot{B}_{\beta_\xi} \subseteq^* \dot{B}_{\beta_\zeta} \text{ for } \zeta < \xi$$

and

$$\Vdash_{\mathbb{P}_{\lambda\cdot\kappa}} \dot{B}_{\beta_\xi} \text{ has infinite intersection with every member of } \mathcal{A}.$$

To verify that the last condition holds, it suffices to use (4) and the fact that $\beta_\xi \notin S$.

Since $\Vdash_{\mathbb{P}_{\lambda\cdot\kappa}} \bigcup_{\xi<\kappa} U_\xi = \iota_*$, we conclude by (4) and the definition of $\dot{\mathbb{Q}}_{\beta_\xi}$ that, in $V^{\mathbb{P}_{\lambda\cdot\kappa}}$, the set $\{B_{\beta_\xi} : \xi < \kappa\}$ is a linear refinement of $\{A_\iota : \iota < \iota_*\}$. ∎

Lemma 3.7. *If $U \in [\lambda]^\mu$ is from the ground model V and $\gamma \in [\lambda, \lambda\cdot\kappa]$, then*

(∗) $\Vdash_{\mathbb{P}_\gamma} |\{\alpha \in U : \dot{C} \subseteq^* \dot{B}_\alpha\}| < \mu$ *for each infinite* $\dot{C} \subseteq \omega$.

Proof. We prove the fact by induction on $\gamma \in [\lambda, \lambda\cdot\kappa]$. For each γ let G_γ denote the $\mathbb{P}_\gamma$-generic filter.

Assume that $\gamma = \lambda$. Let $C \in V[G_\lambda]$ be an infinite subset of ω and let $\dot{C}$ be a $\mathbb{P}_\lambda$-name for C. As C is determined by countably many Cohen reals, we may assume by changing the order that $C \in V[G_\omega]$. Then

$$\Vdash_{\mathbb{P}_\gamma} |\{\alpha \in U : \dot{C} \subseteq^* \dot{B}_\alpha\}| < \aleph_1.$$

We next establish the preservation of (∗) through the steps of iteration.

Assume that $\gamma = \beta + 1$ is a successor ordinal. We will work in $V[G_\beta]$ and force with $\mathbb{Q}_\beta$. Assume that in $V[G_\beta]$, for every infinite C,

(∗∗) $|\{\alpha \in U : C \subseteq^* B_\alpha\}| < \mu$.

We force with the $\mathcal{F}_\beta$-Mathias forcing $\mathbb{Q}_\beta$, where $\mathcal{F}_\beta$ is generated by a centered family of cardinality $< \mu$. Therefore $\mathbb{Q}_\beta$ contains a dense subset D of cardinality $< \mu$. Assume that

$$\Vdash_{\mathbb{Q}_\beta} |\{\alpha \in U : \dot{C} \subseteq^* B_\alpha\}| \geq \mu \text{ for some infinite } \dot{C} \subseteq \omega.$$

Let

$$W = \{\alpha \in U : \exists q_\alpha,\, q_\alpha \Vdash \dot{C} \subseteq^* B_\alpha\}.$$

The set W belongs to the model $V[G_\beta]$. We may assume that $q_\alpha \in D$ for each $\alpha \in W$. By the pigeonhole principle there is $q_* \in D$ and a set $W_1 \subseteq W$ of cardinality μ such that $q_\alpha = q_*$ for each $\alpha \in W_1$. This means that for each $\alpha \in W_1$,

$$q_* \Vdash \dot{C} \subseteq^* B_\alpha.$$

For each $\alpha \in W_1$ there are $r_\alpha \leq q_*$ and k_α such that for each $\alpha \in W_1$,

$$r_\alpha \Vdash \dot{C} \setminus [0, k_\alpha) \subseteq B_\alpha.$$

Again, by the pigeonhole principle there are r_* and k_* and $W_2 \subseteq W_1$ of cardinality μ such that $r_\alpha = r_*$ and $k_\alpha = k_*$ for each $\alpha \in W_2$. This means that for each $\alpha \in W_2$,

$$r_* \Vdash \dot{C} \setminus [0, k_*) \subseteq B_\alpha.$$

It follows that

$$r_* \Vdash \bigcap_{\alpha \in W_2} B_\alpha \text{ is infinite.}$$

But $\bigcap_{\alpha \in W_2} B_\alpha$ belongs to $V[G_\beta]$, contradicting $(**)$.

Assume that γ is a limit ordinal of uncountable cofinality. Let $C \in V[G_\lambda]$ be an infinite subset of ω and let $\dot{C}$ be a $\mathbb{P}_\gamma$-name for C. By [3, Lemma 16.14] there is $\beta < \gamma$ such that $C \in V[G_\beta]$ and $\beta \geq \lambda$. By the inductive hypothesis, we have

$$\Vdash_{\mathbb{P}_\beta} |\{\alpha \in U : \dot{C} \subseteq^* \dot{B}_\alpha\}| < \mu.$$

Since c.c.c. forcing notions preserve cardinality, we deduce that

$$\Vdash_{\mathbb{P}_\gamma} |\{\alpha \in U : \dot{C} \subseteq^* \dot{B}_\alpha\}| < \mu.$$

Finally, let $\gamma > \lambda$ be a limit ordinal with countable cofinality. Fix a sequence $\langle \gamma_n : n < \omega \rangle$ increasing to γ. Towards a contradiction assume that there is $p \in \mathbb{P}_\gamma$ such that

$$p \Vdash_{\mathbb{P}_\gamma} \dot{\mathcal{U}} = \{\alpha \in U : \dot{C} \subseteq^* \dot{B}_\alpha\} \text{ has cardinality } \mu.$$

Let $\dot{\beta}_\iota$ be a name for the ιth element of $\dot{\mathcal{U}}$. Let G be a $\mathbb{P}_\gamma$-generic filter containing p, and for every $\iota < \mu$ let $p_\iota \in G_\gamma$, $\alpha_\iota < \mu$ and $k_\iota < \omega$ be such that:

(a) $p_\iota \leq p$;

(b) $p_\iota \Vdash \dot{\beta}_\iota = \alpha_\iota$;

(c) $p_\iota \Vdash \dot{C} \setminus \dot{B}_{\alpha_\iota} \subseteq [0, k_\iota)$.

As $\operatorname{supp}(p_\iota)$ is finite for each $\iota < \mu$, there exists $n_\iota < \omega$ with $\operatorname{supp}(p_\iota) \subseteq \gamma_{n_\iota}$. Since there are μ many indices ι and only countably many n_ι and k_ι, there exist n_* and k_* such that the set

$$W = \{\iota < \mu : n_\iota = n_*,\ k_\iota = k_*\}$$

has cardinality μ. In particular $p_\iota \in G_{\gamma_{n*}}$ for all $\iota \in W$. Notice that W, $\langle p_\iota : \iota \in W\rangle \in V[G]$ (in fact they belong to $V[G_{\gamma_{n*}}]$). Let $\dot{D}$ be a $\mathbb{P}_{\gamma_{n*}}$-name defined as follows: given a $\mathbb{P}_{\gamma_{n*}}$-filter H, let $\dot{D}[H]$ be a set

$$\{k \in \omega : \exists q \in \mathbb{P}_{\gamma_{n*},\gamma},\ q \Vdash k \in \dot{C}'\};$$

then $\dot{C}'$ is the $\mathbb{P}_{\gamma_{n*},\gamma}$-name obtained in a standard way by "partially evaluating $\dot{C}$ with H". We claim that $p_\iota \Vdash \dot{D} \setminus \dot{B}_{\alpha_\iota} \subseteq [0, k_*)$ for all ι.

Indeed, otherwise there exists $r \le p_i$, $r \in \mathbb{P}_{\gamma_{n*}}$, and $k > k_*$ such that $r \Vdash_{\mathbb{P}_{\gamma_{n*}}} k \in \dot{D} \setminus \dot{B}_{\alpha_\iota}$. Thus

$$r \Vdash_{\mathbb{P}_{\gamma_{n*}}} (\exists q \in \mathbb{P}_{\gamma_{n*},\gamma},\ q \Vdash_{\mathbb{P}_{\gamma_{n*},\gamma}} k \in \dot{C}' \setminus \dot{B}_{\alpha_\iota}).$$

Let $r' \le r$, $r' \in \mathbb{P}_{\gamma_{n*}}$, and $q \in \mathbb{P}_{\gamma_{n*},\gamma}$ be such that

$$r \Vdash_{\mathbb{P}_{\gamma_{n*}}} (q \Vdash_{\mathbb{P}_{\gamma_{n*},\gamma}} k \in \dot{C}' \setminus \dot{B}_{\alpha_\iota}).$$

This means that $r'^\frown q \Vdash_{\mathbb{P}_\gamma} k \in \dot{C} \setminus \dot{B}_{\alpha_\iota}$. But this is impossible because

$$r'^\frown q \le p_\iota \Vdash_{\mathbb{P}_\gamma} \dot{C} \setminus \dot{B}_{\alpha_\iota} \subseteq k_\iota.$$

This proves the claim.

As $p_\iota \in G_{\gamma_{n*}}$, the above claim implies that, in $V[G_{\gamma_{n*}}]$,

$$W \subseteq \{\iota : \dot{D}[G_{\gamma_{n*}}] \subseteq \dot{B}_{\alpha_\iota}[G_{\gamma_{n*}}]\},$$

contradicting the inductive hypothesis. ∎

Lemma 3.8. $\Vdash_{\mathbb{P}_{\lambda\cdot\kappa}} \mathfrak{lr} \le \mu$.

Proof. Assume otherwise. Then there is $p \in \mathbb{P}$ such that $p \Vdash \mu < \mathfrak{lr}$. Take a generic filter G containing p. We argue in $V[G]$. The family $\mathcal{B} = \{B_\alpha : \alpha < \mu\}$ has a linear refinement. By Lemma 2.5, there is a tower $\{T_\iota : \iota < \mathfrak{p}\}$ refining $\mathcal{B}$. Let $D_\iota = \{\alpha : T_\iota \subseteq^* B_\alpha\}$. Since $\bigcup_{\iota<\mathfrak{p}} D_\iota = \mu$, there is ι such that $|D_\iota| = \mu$. This means that $T_\iota \subseteq^* B_\alpha$ for μ many α, contradicting $(*)$. ∎

This completes the proof of Theorem 3.3. ∎

3.3. A model for $\mathfrak{lr} \ll \mathfrak{b} = \mathfrak{lx} = \mathfrak{d} \ll \mathfrak{c}$. Our model will be constructed using Mathias-type forcing notions as in the previous section, together with Hechler forcing.

THEOREM 3.9. *Assume the Generalized Continuum Hypothesis, and let κ, η and λ be uncountable cardinal numbers such that κ and η are regular and $\kappa < \eta < \lambda = \lambda^{<\kappa}$. There is a c.c.c. forcing notion $\mathbb{P}$ of cardinality λ such that*

$$\Vdash_{\mathbb{P}} \mathfrak{p} = \kappa < \mathfrak{lr} = \kappa^+ \le \mathfrak{b} = \mathfrak{l}\mathfrak{x} = \mathfrak{d} = \eta < \lambda = \mathfrak{c}.$$

Proof. We use the iteration of Theorem 3.3, but κ, η, and λ here stand for μ, κ, and λ there, respectively, and we intersperse Hechler's forcing during the iteration. More precisely, the forcing notion $\mathbb{P}$ is given by the following iteration:

(1) $\langle \mathbb{P}_\alpha, \dot{\mathbb{Q}}_\beta : \alpha \le \lambda\cdot\eta, \beta < \lambda\cdot\eta\rangle$ is a finite support iteration of length $\lambda\cdot\eta$ (ordinal product);
(2) $\mathbb{P} = \mathbb{P}_{\lambda\cdot\eta}$;
(3) $\mathbb{P}_0$ is the trivial forcing;
(4) if $\alpha \in \{\lambda\cdot\xi : \xi > 0\}$, then $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is Hechler's forcing;
(5) if $\alpha < \lambda\cdot\eta$ and $\alpha \notin \{\lambda\cdot\xi : \xi > 0\}$, then:
   (a) $\Vdash_{\mathbb{P}_\alpha} \dot{\mathbb{Q}}_\alpha$ is an $\dot{\mathcal{F}}_\alpha$-Mathias forcing;
   (b) $\dot{\mathcal{F}}_\alpha$ is a name for a filter generated by a centered family $\{\dot{A}_{\alpha,\iota} : \iota < \iota_\alpha\}$ which contains cofinite sets, where ι_α is an ordinal $< \kappa$;
   (c) $\iota_\alpha = 0$ for $\alpha < \lambda$ (thus $\mathbb{Q}_\alpha$ is isomorphic to Cohen's forcing for $\alpha < \lambda$);
   (d) $\dot{A}_{\alpha,\iota}$ is a $\mathbb{P}_\alpha$-name for a subset of ω;
   (e) $b_{\alpha,\iota} \colon (2^\omega)^\omega \to [\omega]^\omega$ is a Borel function coded in the ground model;
   (f) $\Vdash_{\mathbb{P}_\alpha} \dot{A}_{\alpha,\iota} = b_{\alpha,\iota}(\langle \dot{B}_{\gamma(\alpha,\iota,n)} : n < \omega\rangle)$, where $B_\alpha \subseteq [\omega]^\omega$ denotes the αth generic real;
   (g) if $\alpha = \lambda\cdot\xi + \nu$ (where $\nu < \lambda$), then
$$\gamma(\alpha,\iota,n) < \lambda\cdot\xi;$$
   (h) for each $\zeta < \xi$ and each sequence $\langle b_\iota : \iota < \iota_*\rangle$ of Borel functions $b_\iota \colon (2^\omega)^\omega \to [\omega]^\omega$ of length $\iota_* < \kappa$, and all ordinal numbers $\delta(\iota,n) < \lambda\cdot\zeta$ such that $\mathbb{P}$ forces that the filter generated by the cofinite sets together with the family
$$\{b_\iota(\langle \dot{B}_{\delta(\iota,n)} : n < \omega\rangle) : \iota < \iota_*\}$$
   is proper, there are arbitrarily large $\alpha < \lambda\cdot(\zeta+1)$ such that:
   - $\iota_\alpha = \iota_*$;
   - $b_{\alpha,\iota} = b_\iota$ for all $\iota < \iota_*$;
   - $\gamma(\alpha,\iota,n) = \delta(\iota,n)$ for all $\iota < \iota_*$ and all n.

Observe that $\Vdash_{\mathbb{P}} \mathfrak{b} = \mathfrak{l}\mathfrak{x} = \mathfrak{d} = \eta$ since Hechler reals are added in steps $\lambda\cdot\xi$ ($\xi < \eta$) of the iteration. Also, $\Vdash_{\mathbb{P}} 2^{\aleph_0} = \lambda$, since $\lambda = \lambda^{<\kappa}$. It remains to prove that $\Vdash_{\mathbb{P}} \mathfrak{p} = \kappa$ and $\Vdash_{\mathbb{P}} \mathfrak{lr} = \kappa^+$.

LEMMA 3.10. *If $U \in [\lambda]^\kappa$ is from the ground model V and $\gamma \in [\lambda, \lambda \cdot \kappa]$, then*

($*$) $\Vdash_{\mathbb{P}_\gamma} |\{\alpha \in U : \dot{C} \subseteq^* \dot{B}_\alpha\}| < \kappa$ *for each infinite* $\dot{C} \subseteq \omega$.

Proof. The proof is as in Lemma 3.7, with one more case to check: $\gamma = \beta + 1$ and $\Vdash_{\mathbb{P}_\beta} \dot{\mathbb{Q}}$ is Hechler's forcing.

In V, enumerate U as $\{\alpha_\delta : \delta < \kappa\}$. Consider a family $\{B_\alpha : \alpha \in U\} = \{B_{\alpha_\delta} : \delta < \kappa\}$ in $V[G_\beta]$. It is *eventually narrow*, that is, for each $C \in [\omega]^\omega$ there is δ_0 such that $C \not\subseteq^* B_{\alpha_\delta}$ for each $\delta > \delta_0$. By [1, Theorem 3.1], eventually narrow families are preserved by Hechler's forcing. Thus,

$$\Vdash_{\mathbb{P}_\gamma} |\{\alpha \in U : \dot{C} \subseteq^* \dot{B}_\alpha\}| < \kappa \text{ for each infinite } \dot{C} \subseteq \omega. \ \blacksquare$$

LEMMA 3.11. $\Vdash_{\mathbb{P}} \kappa \leq \mathfrak{p}$.

Proof. The proof is as in Lemma 3.5. ■

LEMMA 3.12. $\Vdash_{\mathbb{P}} \mathfrak{p} \leq \kappa$.

Proof. By Lemma 3.10 for $U = \kappa$, the family $\{B_\alpha : \alpha \leq \kappa\}$ of the first κ Cohen reals is an example of a centered family in $V[G]$ that has no pseudointersection. ■

LEMMA 3.13. $\Vdash_{\mathbb{P}} \mathfrak{lr} \leq \kappa^+$.

Proof. The proof is as in Lemma 3.8. The only difference is that since now $\kappa = \mu = \mathfrak{p}$, we need to change κ to κ^+ in the conclusion. ■

Consider the following weak version of the Martin's Axiom $M(\kappa)$: If

- $\mathcal{A} \subseteq [\omega]^\omega$ is a centered family of cardinality $< \kappa$ (where $\kappa > \omega$) that contains all cofinite sets;
- $\mathbb{Q} = \mathbb{Q}_{\mathcal{A}}$ is the $\mathcal{A}$-Mathias forcing notion;
- $\mathcal{D}_\beta$ is an open dense subset of $\mathbb{Q}$ for each $\beta < \kappa$,

then there is a filter $H \subseteq \mathbb{Q}$ such that $H \cap \mathcal{D}_\beta \neq \emptyset$ for each $\beta < \kappa$.

LEMMA 3.14. *$M(\kappa)$ implies that $\mathfrak{lr} \geq \kappa^+$.*

Proof. Assume that $\{A_\alpha : \alpha < \kappa\}$ is a centered family. We may further assume that it contains all cofinite sets and is closed under finite intersections. We choose A_α^- by induction on $\alpha < \kappa$ such that:

- $A_\alpha^- \subseteq^* A_\beta^-$ for each $\beta < \alpha$;
- $A_\alpha^- \cap A_\beta$ is infinite for each $\beta < \kappa$.

Assume that A_β^- is defined for $\beta < \alpha$. Let $\mathcal{A}$ be the closure of the family $\{A_\beta^- : \beta < \alpha\}$ under finite intersections and cofinite sets. Apply $M(\kappa)$ to the family $\mathcal{A}$ and the dense sets $\mathcal{D}_{\beta,k} = \{\langle u, B\rangle : |u \cap A_\beta| \geq k\}$ (where $\beta < \kappa$ and $n < \omega$) to obtain H. Set $A_\alpha^- = \bigcup\{u : \langle u, B\rangle \in H\}$. ■

LEMMA 3.15. $\Vdash_{\mathbb{P}} \mathfrak{lr} \geq \kappa^+$.

Proof. Assume that $\mathcal{A}$ is a centered family of cardinality $< \kappa$ in the extended model $V[G]$, which contains all cofinite sets, and $\mathscr{D}$ is a family of $\leq \kappa$ open dense subsets of $\mathbb{Q} = \mathbb{Q}_{\mathcal{A}}$. Assume that p forces that $\dot{\mathcal{A}} = \{\dot{A}_\iota : \iota < \dot{\iota_*} < \kappa\}$ and $\dot{\mathscr{D}} = \{\dot{\mathcal{D}}_\epsilon : \epsilon < \kappa\}$ form a counterexample. The forcing is c.c.c., and p forces that $\dot{\iota_*} < \kappa$. We may assume that ι_* is in the ground model.

As $\eta = \operatorname{cof}(\eta) > \kappa$, we may assume that all $\dot{A}_\iota, \dot{\mathcal{D}}_\epsilon$ are $\mathbb{P}_{\lambda\cdot\xi}$-names for some $\xi < \eta$. We can find $\alpha \in [\lambda\cdot\xi, \lambda\cdot(\xi+1))$ such that $\{\dot{A}_{\alpha,\iota} : \iota < \iota_\alpha\} = \{\dot{A}_\iota : \iota < \iota_*\}$ is forced. We conclude as in the proof of the consistency of Martin's Axiom. ∎

The proof of Theorem 3.9 is completed. ∎

4. Open problems. One of our main results (Theorem 2.10) is that the cofinality of $\mathfrak{lr}$ is uncountable.

PROBLEM 4.1. *Is it consistent that* $\mathfrak{lr}$ *is singular?*

We introduce below two ad-hoc names for combinatorial cardinal characteristics. Once progress is made on the associated problems, better names may be introduced.

DEFINITION 4.2. Let κ_1 be the minimal cardinality of a family $\mathcal{A} \subseteq ([\omega]^\omega)^\omega$ such that:

- for each n, $\{A(n) : A \in \mathcal{A}\}$ is linear;
- there is *no* $g \in \omega^\omega$ such that the sets $S_A := \{n : g(n) \in A(n)\}$ are infinite, and the family $\{S_A : A \in \mathcal{A}\}$ has a linear refinement.

The following assertions are proved exactly as in Section 2.2.

LEMMA 4.3.

- $\operatorname{non}(\mathsf{S}_1(\mathrm{T}^*, \mathrm{T}^*)) = \kappa_1$;
- $\min\{\mathfrak{b}, \mathfrak{s}, \operatorname{cov}(\mathcal{M})\} \leq \kappa_1$;
- $\min\{\max\{\mathfrak{lr}, \min\{\mathfrak{b}, \mathfrak{s}\}\}, \operatorname{cov}(\mathcal{M})\} \leq \kappa_1$. ∎

PROBLEM 4.4. *Can we express* κ_1*, in ZFC, in terms of classical combinatorial cardinal characteristics of the continuum?*

DEFINITION 4.5. Let κ_{fin} be the minimal cardinality of a family $\mathcal{A} \subseteq ([\omega]^\omega)^\omega$ such that:

(1) for each n, $\{A(n) : A \in \mathcal{A}\}$ is linear;
(2) there are *no* finite sets $F_0, F_1, \ldots \subseteq \omega$ such that the sets $S_A := \bigcup_n \{n\} \times (A(n) \cap F_n) \subseteq \omega \times \omega$ are infinite, and the family $\{S_A : A \in \mathcal{A}\}$ has a linear refinement.

We have the following.

Lemma 4.6.

(1) $\mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{T}^*, \mathrm{T}^*)) = \mathrm{non}(\mathsf{S}_{\mathrm{fin}}(\mathrm{T}, \mathrm{T}^*)) = \kappa_{\mathrm{fin}}$;

(2) $\min\{\mathrm{cov}(\mathcal{M}), \mathfrak{lr}\}, \min\{\mathfrak{b}, \mathfrak{s}\} \leq \kappa_{\mathrm{fin}} \leq \mathfrak{l}\mathfrak{x}$. ■

Problem 4.7. *Can we express κ_{fin}, in ZFC, in terms of classical combinatorial cardinal characteristics of the continuum? If not, can we improve the above bounds in ZFC?*

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