o-BOUNDED GROUPS AND OTHER TOPOLOGICAL GROUPS WITH STRONG COMBINATORIAL PROPERTIES

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ABSTRACT. We construct several topological groups with very strong combinatorial properties. In particular, we give simple examples of subgroups of $\mathbb{R}$ (thus strictly o-bounded) which have the Menger and Hurewicz properties but are not σ-compact, and show that the product of two o-bounded subgroups of $\mathbb{R}^{\mathbb{N}}$ may fail to be o-bounded, even when they satisfy the stronger property $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. This solves a problem of Tkačenko and Hernandez, and extends independent solutions of Krawczyk and Michalewski and of Banakh, Nickolas, and Sanchis. We also construct separable metrizable groups G of size continuum such that every countable Borel ω-cover of G contains a γ-cover of G.

1. Introduction

In [15, 11], a unified framework for topological diagonalizations is established, which turns out to be closely related to several notions which appear in a recently flourishing study of topological groups in terms of their covering properties (see, e.g., [18, 9, 10, 12, 13] and references therein). A comprehensive study of these interrelations is currently being carried out by Babinkostova, Kočinac, and Scheepers [1]. The purpose of this paper is to adopt several recent construction techniques from the general theory of topological diagonalizations to the theory of topological groups.

1.1. **Topological diagonalizations.** We briefly describe the general framework. Let X be a topological space. An open cover $\mathcal{U}$ of X is an ω-*cover* of X if X is not in $\mathcal{U}$, and for each finite subset F of X, there is a set $U \in \mathcal{U}$ such that $F \subseteq U$. $\mathcal{U}$ is a γ-*cover* of X if it is infinite and for each x in X, $x \in U$ for all but finitely many $U \in \mathcal{U}$. Let $\mathcal{O}$, Ω, and Γ denote the collections of all countable open covers, ω-covers, and γ-covers of X, respectively. Let $\mathcal{A}$ and $\mathcal{B}$ be collections of covers of a space X. Following are selection hypotheses which X might satisfy or not satisfy:

$\mathsf{S}_1(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathcal{A}$, there exist members $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{U_n\}_{n\in\mathbb{N}} \in \mathcal{B}$.

$\mathsf{S}_{fin}(\mathcal{A}, \mathcal{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathcal{A}$, there exist finite (possibly empty) subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\bigcup_{n\in\mathbb{N}} \mathcal{F}_n \in \mathcal{B}$.

$\mathsf{U}_{fin}(\mathcal{A},\mathcal{B})$: For each sequence $\{\mathcal{U}_n\}_{n\in\mathbb{N}}$ of members of $\mathcal{A}$ which do not contain a finite subcover, there exist finite (possibly empty) subsets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $\{\bigcup \mathcal{F}_n\}_{n\in\mathbb{N}} \in \mathcal{B}$.

Some of the properties defined in this manner were studied earlier by Hurewicz ($\mathsf{U}_{fin}(\mathcal{O},\Gamma)$), Menger ($\mathsf{S}_{fin}(\mathcal{O},\mathcal{O})$), Rothberger ($\mathsf{S}_1(\mathcal{O},\mathcal{O})$, traditionally known as the C'' property), Gerlits and Nagy ($\mathsf{S}_1(\Omega,\Gamma)$, traditionally known as the γ-property), and others. Many equivalences hold among these properties, and the surviving ones appear in Figure 1 (where an arrow denotes implication), to which no arrow can be added except perhaps from $\mathsf{U}_{fin}(\mathcal{O},\Gamma)$ or $\mathsf{U}_{fin}(\mathcal{O},\Omega)$ to $\mathsf{S}_{fin}(\Gamma,\Omega)$ [11].
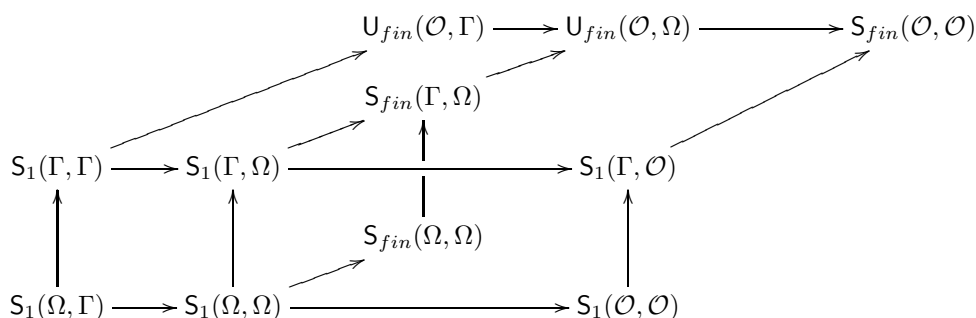

FIGURE 1. The Scheepers Diagram

Each selection principle has a naturally associated game, but we will restrict our attention to the game $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$, which is played as follows: In the nth inning, ONE chooses an element $\mathcal{U}_n$ of $\mathcal{A}$ and then TWO responds by choosing a finite subset $\mathcal{F}_n$ of $\mathcal{U}_n$. They play an inning per natural number. A play $(\mathcal{U}_0,\mathcal{F}_0,\mathcal{U}_1,\mathcal{F}_1,\dots)$ is won by TWO if $\bigcup_n \mathcal{F}_n \in \mathcal{B}$; otherwise ONE wins. We will write ONE $\uparrow \mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$ (respectively, TWO $\uparrow \mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$) for "ONE (respectively, TWO) has a winning strategy in $\mathsf{G}_{fin}(\mathcal{A},\mathcal{B})$".

1.2. o-**bounded groups.** Okunev introduced the following notion as an approximation of σ-compact groups: A topological group G is o-bounded if for each sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of the unit element of G, there exists a sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of G such that $G = \bigcup_n F_n \cdot U_n$. It is possible to state this definition in the language of selection principles. Let $\mathcal{O}_{\mathsf{nbd}}$ denote the covers of G of the form $\{g \cdot U : g \in G\}$, where U is a neighborhood of the unit element of G. Then G is o-bounded if, and only if, G satisfies $\mathsf{S}_{fin}(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$.

According to Tkačenko, a topological group G is *strictly o-bounded* if TWO has a winning strategy in the game $\mathsf{G}_{fin}(\mathcal{O}_{\mathsf{nbd}},\mathcal{O})$. Clearly, every subgroup of a σ-compact group is strictly o-bounded, but the converse does not hold [9].

Notational convention. For sets X,Y, ^XY denotes the collection of all functions from X to Y. If Y is a topological space, then the topology on ^XY is the Tychonoff product topology.

2. Two almost σ-compact subgroups of $\mathbb{R}$

The *Baire space* $^{\mathbb{N}}\mathbb{N}$ is (quasi)ordered by eventual dominance: $f \leq^* g$ if $f(n) \leq g(n)$ for all but finitely many n. A subset of $^{\mathbb{N}}\mathbb{N}$ is *dominating* if it is cofinal in $^{\mathbb{N}}\mathbb{N}$ with respect to $\leq^*$. If a subset of $^{\mathbb{N}}\mathbb{N}$ is unbounded with respect to $\leq^*$, then we simply say that it is *unbounded*. Let $\mathfrak{b}$ (respectively, $\mathfrak{d}$) denote the minimal cardinality of an unbounded (respectively, dominating) subset of $^{\mathbb{N}}\mathbb{N}$.

We use the following setting from [5]. Let $\overline{\mathbb{N}} = \mathbb{N} \cup \{\infty\}$ be the one point compactification of $\mathbb{N}$. (A subset $A \subseteq \overline{\mathbb{N}}$ is open if $A \subseteq \mathbb{N}$, or $\infty \in A$ and A is cofinite.) Let $^{\mathbb{N}}\nearrow\overline{\mathbb{N}} \subseteq {}^{\mathbb{N}}\overline{\mathbb{N}}$ consist of the *nondecreasing* functions $f \in {}^{\mathbb{N}}\overline{\mathbb{N}}$. $^{\mathbb{N}}\nearrow\overline{\mathbb{N}}$ is homeomorphic to the Cantor set of reals. For each increasing finite sequence s of natural numbers, let $q_s \in {}^{\mathbb{N}}\nearrow\overline{\mathbb{N}}$ be defined by $q_s(k) = s(k)$ if $k < |s|$, and $q_s(k) = \infty$ otherwise. Let Q be the collection of all these elements q_s.

Theorem 1. *There exists a non-σ-compact subgroup G_H of $\mathbb{R}$ of cardinality $\mathfrak{b}$ such that all finite powers of G_H satisfy $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$ (in particular, they satisfy $\mathsf{S}_{fin}(\Omega, \Omega)$).*

Proof. Let $B = \{f_\alpha : \alpha < \mathfrak{b}\} \subseteq {}^{\mathbb{N}}\mathbb{N}$ be a $\leq^*$-unbounded set of strictly increasing elements of $^{\mathbb{N}}\mathbb{N}$ which forms a $\mathfrak{b}$-*scale* (that is, for each $\alpha < \beta$, $f_\alpha \leq^* f_\beta$), and set $H = B \cup Q$. In [5] it is proved that all finite powers of H satisfy $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$.

Think of H as a set of real numbers. For each n, the set

$$G_n = \{\alpha_1 g_1 + \cdots + \alpha_n g_n : \alpha_1, \ldots, \alpha_n \in \mathbb{Z}, \ g_1, \ldots, g_n \in H\}$$

is a union of countably many continuous images of H^n, thus for each k, $(G_n)^k$ is a union of countably many continuous images of H^{nk}. As the property $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$ is preserved under taking continuous images and countable unions [11, 4], we have that each set $(G_n)^k$ satisfies $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$.

Take $G_H = \langle H \rangle$. Then $G_H = \bigcup_{n\in\mathbb{N}} G_n$ and for each n, $G_n \subseteq G_{n+1}$. Thus, $(G_H)^k = \bigcup_{n\in\mathbb{N}}(G_n)^k$ for each k; therefore $(G_H)^k$ satisfies $\mathsf{U}_{fin}(\mathcal{O}, \Gamma)$ for each k. Now, satisfying $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$ in all finite powers implies $\mathsf{S}_{fin}(\Omega, \Omega)$ [11].

It was observed by Pol and Zdomsky that one can make sure that G_H is not σ-compact by embedding $^{\mathbb{N}}\nearrow\overline{\mathbb{N}}$ in a Cantor set of reals, C, that is *linearly independent* over $\mathbb{Q}$. This way, $H = \langle H \rangle \cap C$ is a closed subset of $\langle H \rangle = G_H$. Since H is not σ-compact [5], we get that G_H cannot be σ-compact. $\square$

Problem 2. *Does G_H satisfy $\mathsf{S}_1(\Gamma, \Gamma)$?*

Corollary 3. *TWO $\uparrow$ $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$ is strictly stronger than TWO $\uparrow$ $\mathsf{G}_{fin}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ (strict o-boundedness).*

Proof. By a well-known theorem of Telgársky, TWO $\uparrow$ $\mathsf{G}_{fin}(\mathcal{O}, \mathcal{O})$ if, and only if, the space G is σ-compact. Contrast this with Theorem 1. $\square$

Theorem 1 has a group-theoretic consequence.

Corollary 4. *Assume that for each m, $\{g_{m,n} + U_m\}_{n\in\mathbb{N}}$ is a cover of G_H. Then there exists a sequence $\{m_n\}_{n\in\mathbb{N}}$ such that*

$$G_H = \bigcup_k \bigcap_{n>k} (\{g_{n,1}, \ldots, g_{n,m_n}\} + U_n).$$

In fact, this property is satisfied by all finite powers of G_H.

Let $D = \{g_\alpha : \alpha < \mathfrak{d}\}$ be a dominating subset of $^{\mathbb{N}}\mathbb{N}$ where each g_α is increasing, and for each $f \in {}^{\mathbb{N}}\mathbb{N}$ there exists $\alpha_0 < \mathfrak{d}$ such that for any finite set $F \subseteq \mathfrak{d} \setminus \alpha_0$, $f(n) < \min\{g_\beta(n) : \beta \in F\}$ for infinitely many n. Such a set was constructed in [5].

A subset F of $^{\mathbb{N}}\mathbb{N}$ is *finitely-dominating* if for each $g \in {}^{\mathbb{N}}\mathbb{N}$ there exist k and $f_1, \ldots, f_k \in {}^{\mathbb{N}}\mathbb{N}$ such that $g(n) \leq^* \max\{f_1(n), \ldots, f_k(n)\}$. For conciseness, we use the following shortened notation:

Ax: Either a union of less than $\mathfrak{d}$ many not dominating sets is not dominating (in other words, $\mathfrak{b} = \mathfrak{d}$), or else a union of less than $\mathfrak{d}$ many not finitely-dominating sets of increasing functions is not finitely-dominating.

In [5] it is shown that Ax implies that $M = D \cup Q$ satisfies $\mathsf{S}_{fin}(\Omega, \Omega)$. (Observe that $\mathsf{S}_{fin}(\Omega, \Omega)$ is preserved under taking finite powers [11].) Assuming Ax, one shows as in Theorem 1 that all finite powers of $G_M = \langle M \rangle$ satisfy $\mathsf{U}_{fin}(\mathcal{O}, \mathcal{O})$ and obtains the following.

Theorem 5. *Assume that* Ax *holds. Then:*

(1) *There exists a non-σ-compact subgroup G_M of $\mathbb{R}$ of cardinality $\mathfrak{d}$ such that G_M satisfies $\mathsf{S}_{fin}(\Omega, \Omega)$.*

(2) *Assume that for each m, $\{g_{m,n} + U_m\}_{n \in \mathbb{N}}$ is a cover of G_M. Then there exists a sequence $\{m_n\}_{n \in \mathbb{N}}$ such that for each finite subset F of G_M, there exists n such that $F \subseteq \{g_{n,1}, \ldots, g_{n,m_n}\} + U_n$. Moreover, this holds for all finite powers of G_M.*

It follows from the next section that the hypothesis Ax is not necessary to prove Theorem 5 (namely, it also follows from the incomparable assumption $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$).

Problem 6. *Is Theorem* 5 *provable in ZFC?*

3. Products of o-bounded groups

In Problem 3.2 of [18] and Problem 5.2 of [9] it is asked whether the (Tychonoff) product of two o-bounded groups is o-bounded. We give a negative answer. A negative answer was independently given by Krawczyk and Michalewski [13], but our result is stronger in the following sense: Let G be a topological group such that all finite powers of G are Lindelöf. Then each open ω-cover of G contains a *countable* ω-cover of G. Let $\mathcal{B}_\Omega$ denote the collection of all *countable Borel ω-covers* of G. In this case,

$$\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega) \to \mathsf{S}_1(\Omega, \Omega) \to \mathsf{S}_{fin}(\mathcal{O}, \mathcal{O}),$$

where no implication can be reversed [11, 17], and the last property (the Menger property) implies $\mathsf{S}_{fin}(\mathcal{O}_{\mathsf{nbd}}, \mathcal{O})$ (o-boundedness). In [13] it is proved that, assuming $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ (this is a small portion of the Continuum Hypothesis), there exist groups G_1 and G_2 satisfying the Menger property $\mathsf{S}_{fin}(\mathcal{O}, \mathcal{O})$, such that $G_1 \times G_2$ is not o-bounded. We use the same hypothesis to show that there exist such groups satisfying $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$.

Fact 7. *Assume that G is a $\leq^*$-dominating subgroup of $^{\mathbb{N}}\mathbb{R}$. Then G is not o-bounded.*

Proof. As $\mathbb{N}$ can be partitioned into infinitely many infinite sets, the following holds.

Lemma 8. *For each o-bounded group G and sequence $\{U_n\}_{n\in\mathbb{N}}$ of neighborhoods of the identity of G, there exists a sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of G with $G = \bigcap_m \bigcup_{n>m} F_n \cdot G$.*

Consider the open sets $U_n = \{f \in {}^{\mathbb{N}}\mathbb{R} : |f(n)| < 1\} \subseteq {}^{\mathbb{N}}\mathbb{R}$. For each sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of ${}^{\mathbb{N}}\mathbb{R}$, $\bigcup_n F_n + U_n$ is $\leq^*$-bounded in ${}^{\mathbb{N}}\mathbb{R}$. Let $h \in {}^{\mathbb{N}}\mathbb{R}$ be a witness for that. As G is dominating, there exists $g \in G$ such that $h \leq^* g$. Then $g \notin \bigcap_m \bigcup_{n>m} F_n \cdot G$. $\qquad\square$

Let $\mathfrak{c} = |\mathbb{R}|$. The assertion $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$ means that $\mathbb{R}$ (or any complete, separable, metric space) is not the union of less than continuum many meager (= first category) sets. We say that L is a κ-*Luzin group* if $|L| \geq \kappa$, and for each meager set M in L, $|L \cap M| < \kappa$.

Theorem 9. *Assume that $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. Then there exist $\mathfrak{c}$-Luzin subgroups L_1, L_2 of ${}^{\mathbb{N}}\mathbb{R}$ satisfying $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$, such that the group $L_1 \times L_2$ is not o-bounded. (These Luzin groups are, in fact, linear vector spaces over $\mathbb{Q}$.)*

Proof. We extend the technique of [17, 4]. We stress that there exists a much easier proof if we only require that L_1 and L_2 satisfy $\mathsf{S}_1(\Omega, \Omega)$; however we do not supply this easier proof to avoid repetitions.

A cover $\mathcal{U}$ of X is ω-*fat* if for each finite $F \subseteq X$ and each finite family $\mathcal{F}$ of nonempty open sets, there exists $U \in \mathcal{U}$ such that $F \subseteq U$ and for each $O \in \mathcal{F}$, $U \cap O$ is not meager. Let $\mathcal{M}$ denote the meager subsets of ${}^{\mathbb{N}}\mathbb{R}$.

Lemma 10 ([4])**.** *Assume that $\mathcal{U}$ is an ω-fat cover of a set $X \subseteq {}^{\mathbb{N}}\mathbb{R}$. Then:*

(1) $\bigcup \mathcal{U}$ *is comeager.*
(2) *For each finite $F \subseteq G$ and finite family $\mathcal{F}$ of nonempty open sets,*

$$\mathcal{U}_{F,\mathcal{F}} := \{U \in \mathcal{U} : F \subseteq U \text{ and for each } O \in \mathcal{F}, U \cap O \notin \mathcal{M}\}$$

is an ω-fat cover of G. Consequently, $\bigcup \mathcal{U}_{F,\mathcal{F}}$ is comeager.

For shortness, we will say that a cover $\mathcal{U}$ of X is *good* if: For each finite $A \subseteq \mathbb{Q} \setminus \{0\}$ and finite $B \subseteq G$, the family

$$\mathcal{U}^{A,B} := \left\{ \bigcap_{q \in A, g \in B} q(U - g) : U \in \mathcal{U} \right\}$$

is an ω-fat cover of X. These covers allow the inductive construction hinted to in the following lemma.

Lemma 11. *Assume that $\mathcal{U}$ is a good cover of a group $G \subseteq {}^{\mathbb{N}}\mathbb{R}$. Then for each element x in the intersection of all sets of the form $\bigcup(\mathcal{U}^{A,B})_{F,\mathcal{F}}$ where the members of $\mathcal{F}$ are* basic *open sets, $\mathcal{U}$ is a good cover of the group $G + \mathbb{Q}x$.*

Proof. Fix finite sets $A \subseteq \mathbb{Q} \setminus \{0\}$ and $B \subseteq G$. We may assume that $1 \in A$. We must show that $\mathcal{U}^{A,B}$ is an ω-fat cover of $G + \mathbb{Q}x$. Let F be a finite subset of $G + \mathbb{Q}x$, and let $\mathcal{F}$ be a finite family of nonempty open sets. By moving to subsets we may assume that all members of $\mathcal{F}$ are basic open sets.

Choose finite sets $\tilde{A} \subseteq \mathbb{Q} \setminus \{0\}$, $\tilde{B} \subseteq G$, and $\tilde{F} \subseteq G$, such that $F \subseteq (\tilde{B} + \tilde{A}x) \cup \tilde{F}$, $1 \in \tilde{A}$, and $0 \in \tilde{B}$. As $x \in \bigcup(\mathcal{U}^{\tilde{A}^{-1}A, B+A^{-1}\tilde{B}})_{\tilde{F}, \mathcal{F}}$, there exists $U \in \mathcal{U}$ such that $x \in \tilde{V} := \bigcap_{q \in \tilde{A}^{-1}A, g \in B+A^{-1}\tilde{B}} q(U - g)$, $\tilde{F} \subseteq \tilde{V}$, and for each $O \in \mathcal{F}$, $\tilde{V} \cap O$

is not meager. Take $V = \bigcap_{q \in A, g \in B} q(U - g)$. Then $\tilde{V} \subseteq V$, thus $\tilde{F} \subseteq V$ and for each $O \in \mathcal{F}$, $V \cap O$ is not meager. Now, $x \in \tilde{V}$, thus for each $\tilde{a} \in \tilde{A}$ and $\tilde{b} \in \tilde{B}$, $x \in \bigcap_{q \in \tilde{a}^{-1}A, g \in B + A^{-1}\tilde{b}} q(U - g)$, thus $x \in \bigcap_{q \in A, g \in B} \tilde{a}^{-1}q(U - (g + q^{-1}\tilde{b}))$, therefore $x \in \bigcap_{q \in A, g \in B} \tilde{a}^{-1}(q(U - g) - \tilde{b})$, thus $\tilde{a}x + \tilde{b} \in V$. This shows that $F \subseteq (\tilde{A}x + \tilde{B}) \cup \tilde{F} \subseteq V \in \mathcal{U}^{A,B}$, and we are done. $\square$

Since we are going to construct Luzin groups, the following lemma tells us that we need not consider covers which are not good.

Lemma 12. *Assume that L is a subgroup of $^{\mathbb{N}}\mathbb{R}$ such that $\mathbb{Q} \cdot L \subseteq L$ and for each nonempty basic open set O, $L \cap O$ is not meager. Then every countable Borel ω-cover $\mathcal{U}$ of L is a good cover of L.*

Proof. Assume that $\mathcal{U}$ is a countable collection of Borel sets which is not a good cover of L. Then there exist finite sets $A \subseteq \mathbb{Q} \setminus \{0\}$, $B \subseteq L$, $F \subseteq L$, and $\mathcal{F}$ of nonempty open sets such that for each $V \in \mathcal{U}^{A,B}$ containing F, $V \cap O$ is meager for some $O \in \mathcal{F}$. For each $O \in \mathcal{F}$ let

$$M_O = \bigcup \{V \in \mathcal{U}^{A,B} : F \subseteq V \text{ and } V \cap O \in \mathcal{M}\}.$$

Then $M_O \cap O$ is meager, thus there exists $x_O \in (L \cap O) \setminus M_O$. Then $F \cup \{x_O : O \in \mathcal{F}\}$ is not covered by any $V \in \mathcal{U}^{A,B}$. We will show that this cannot be the case.

Put $F' = A^{-1}F + B$. As $\mathbb{Q} \cdot L \subseteq L$, F' is a finite subset of L. Thus, there exists $U \in \mathcal{U}$ such that $F' \subseteq U$. Consequently, for each $q \in A$ and $g \in B$, $x \in q(U - g)$ for each $x \in F$, that is, $F \subseteq \bigcap_{q \in A, g \in B} q(U - g) \in \mathcal{U}^{A,B}$. $\square$

We need one more lemma. Denote the collection of countable Borel good covers of a set X by $\mathcal{B}_{\text{good}}$.

Lemma 13. *If $|X| < \mathsf{cov}(\mathcal{M})$, then X satisfies $\mathsf{S}_1(\mathcal{B}_{\text{good}}, \mathcal{B}_{\text{good}})$.*

Proof. Assume that $|X| < \mathsf{cov}(\mathcal{M})$, and let $\{\mathcal{U}_n\}_{n \in \mathbb{N}}$ be a sequence of countable Borel good covers of X. Enumerate each cover $\mathcal{U}_n$ by $\{U_k^n\}_{k \in \mathbb{N}}$. Let $\{Y_n\}_{n \in \mathbb{N}}$ be a partition of $\mathbb{N}$ into infinitely many infinite sets. For each m, let $y_m \in {}^{\mathbb{N}}\mathbb{N}$ be an increasing enumeration of Y_m. Let $\{\mathcal{F}_n\}_{n \in \mathbb{N}}$ be an enumeration of all finite families of nonempty basic open sets.

For finite sets $F, B \subseteq X$ and $A \subseteq \mathbb{Q} \setminus \{0\}$, and each m define a function $\Psi_{F,m}^{A,B} \in {}^{\mathbb{N}}\mathbb{N}$ by

$$\Psi_{F,m}^{A,B}(n) = \min\{k : F \subseteq V := \bigcap_{q \in A, g \in B} q(U_k^{y_m(n)} - g) \text{ and } (\forall O \in \mathcal{F}_m) \, V \cap O \notin \mathcal{M}\}.$$

Since there are less than $\mathsf{cov}(\mathcal{M})$ many functions $\Psi_{F,m}^{A,B}$, there exists by [3] a function $f \in {}^{\mathbb{N}}\mathbb{N}$ such that for each m, F, A, and B, $\Psi_{F,m}^{A,B}(n) = f(n)$ for infinitely many n. Consequently, $\mathcal{V} = \{U_{f(n)}^{y_m(n)} : m, n \in \mathbb{N}\}$ is a good cover of X. $\square$

We are now ready to carry out the construction. Let $^{\mathbb{N}}\mathbb{R} = \{y_\alpha : \alpha < \mathfrak{c}\}$, $\{M_\alpha : \alpha < \mathfrak{c}\}$ be all F_σ meager subsets of $^{\mathbb{N}}\mathbb{R}$, and let $\{\{\mathcal{U}_n^\alpha\}_{n \in \mathbb{N}} : \alpha < \mathfrak{c}\}$ be all sequences of countable families of Borel sets. Let $\{O_k : k \in \mathbb{N}\}$ and $\{\mathcal{F}_m : m \in \mathbb{N}\}$ be all nonempty basic open sets and all finite families of nonempty basic open sets, respectively, in $^{\mathbb{N}}\mathbb{R}$.

We construct L_1 and L_2 by induction on $\alpha < \mathfrak{c}$ as follows. At stage $\alpha \geq 0$ set $L_\alpha^i = \bigcup_{\beta < \alpha} L_\beta^i$ and consider the sequence $\{\mathcal{U}_n^\alpha\}_{n \in \mathbb{N}}$. Say that α is i-good if for

each n, $\mathcal{U}_n^\alpha$ is a good cover of L_α^i. In this case, by Lemma 13 there exist elements $U_n^{\alpha,i} \in \mathcal{U}_n^\alpha$ such that $\mathcal{U}^{\alpha,i} = \{U_n^{\alpha,i}\}_{n\in\mathbb{N}}$ is a good cover of L_α^i. We make the inductive hypothesis that for each i-good $\beta < \alpha$, $\mathcal{U}^{\beta,i}$ is a good cover of L_α^i. For finite sets $F, B \subseteq L_\alpha^i$ and $A \subseteq \mathbb{Q} \setminus \{0\}$, each i-good $\beta \leq \alpha$, and each m define

$$G_{A,B,F,m}^{\beta,i} = \bigcup((\mathcal{U}^{\beta,i})^{A,B})_{F,\mathcal{F}_m}.$$

As $\mathcal{U}^{\beta,i}$ is a good cover of X_α^i, $(\mathcal{U}^{\beta,i})^{A,B}$ is ω-fat cover of L_α^i, and by Lemma 10, $G_{A,B,F,m}^{\beta,i}$ is comeager in $^\mathbb{N}\mathbb{R}$. Set

$$Y_\alpha = \bigcup_{\beta<\alpha} M_\beta \cup \bigcup \left\{ {}^\mathbb{N}\mathbb{R} \setminus G_{A,B,F,m}^{\beta,i} : \begin{array}{c} i < 2, \ i\text{-good } \beta \leq \alpha, m \in \mathbb{N}, \\ \text{finite sets } F, B \subseteq L_\alpha^i, A \subseteq \mathbb{Q} \setminus \{0\} \end{array} \right\},$$

and $Y_\alpha^* = \{x \in {}^\mathbb{N}\mathbb{R} : (\exists y \in Y_\alpha) \ x =^* y\}$ (where $x =^* y$ means that $x(n) = y(n)$ for all but finitely many n). Then Y_α^* is a union of less than $\mathsf{cov}(\mathcal{M})$ many meager sets.

Lemma 14 ([4]). *If X is a union of less than $\mathsf{cov}(\mathcal{M})$ many meager sets in $^\mathbb{N}\mathbb{R}$, then for each $x \in {}^\mathbb{N}\mathbb{R}$ there exist $y, z \in {}^\mathbb{N}\mathbb{R} \setminus X$ such that $y + z = x$.*

Use Lemma 14 to pick $x_\alpha^0, x_\alpha^1 \in {}^\mathbb{N}\mathbb{R} \setminus Y_\alpha^*$ such that $x_\alpha^0 + x_\alpha^1 = y_\alpha$. Let $k = \alpha \bmod \omega$, and change a finite initial segment of x_α^0 and x_α^1 so that they both become members of O_k. Then $x_\alpha^0, x_\alpha^1 \in O_k \setminus Y_\alpha$, and $x_\alpha^0 + x_\alpha^1 =^* y_\alpha$. Finally, define $X_{\alpha+1}^i = L_\alpha^i + \mathbb{Q} \cdot x_\alpha^i$. By Lemma 11, the inductive hypothesis is preserved. This completes the construction.

Take $L_i = \bigcup_{\alpha<\mathfrak{c}} L_\alpha^i$, $i = 1, 2$. By Lemma 13, each L_i satisfies $\mathsf{S}_1(\mathcal{B}_{\text{good}}, \mathcal{B}_{\text{good}})$, and by the construction, its intersection with each nonempty basic open set has size $\mathfrak{c}$. By Lemma 12, $\mathcal{B}_{\text{good}} = \mathcal{B}_\Omega$ for L_i. Finally, $L_1 + L_2$ (a homomorphic image of $L_1 \times L_2$) is a dominating subset of $^\mathbb{N}\mathbb{R}$, thus $L_1 \times L_2$ is not o-bounded. $\qquad\square$

Remark 15. None of the o-bounded groups L_1 and L_2 in Theorem 9 is strictly o-bounded. By an unpublished result of Michalewski, every metrizable strictly o-bounded group is a subgroup of a σ-compact group, and in [9, Theorem 5.3] it is proved that a product of such a group with an o-bounded group is o-bounded.

The following consequence of Theorem 9 seems nontrivial. Say that a subset S of a topological group G is $\aleph_0$-bounding if there exists a countable set $F \subseteq G$ such that $F \cdot S = G$. For example, G is $\aleph_0$-*bounded* if each nonempty open set in G is $\aleph_0$-bounding. The first property in the following corollary may be called *Borel o-boundedness*. This property is more interesting when the group in question is $\aleph_0$-bounded, in which case it is stronger than o-boundedness.

Corollary 16. *Assume that $\mathsf{cov}(\mathcal{M}) = \mathfrak{c}$. Then there exists an $\aleph_0$-bounded topological group G of size continuum such that:*

(1) *For each sequence $\{B_n\}_{n\in\mathbb{N}}$ of $\aleph_0$-bounding Borel sets in G, there exists a sequence $\{F_n\}_{n\in\mathbb{N}}$ of finite subsets of G such that $G = \bigcup_n F_n \cdot B_n$.*

(2) *Moreover, the sequence in (1) will have the property that for each finite $F \subseteq G$ there exists n such that $F \subseteq F_n \cdot U_n$.*

Remark 17. Banakh, Nickolas, and Sanchis have also, independently, proved the consistency of o-bounded groups not being closed under taking finite products (however, they do not consider stronger combinatorial properties as done in [13] and here). Their construction uses ultrafilters which are not nearly coherent; see [2].

4. Groups satisfying $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ or $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$

One may wonder whether Theorem 9 can be strengthened further so that L_1 and L_2 will satisfy a stronger property. By inspection of the Scheepers Diagram 1, the only candidate for a stronger property (among the ones considered in this paper) is $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$. This is far from possible: A result of [1] (see [16, Theorem 32]) implies that whenever G_1 is a topological group satisfying $\mathsf{S}_1(\Omega, \Gamma)$, and G_2 is o-bounded, the group $G_1 \times G_2$ is o-bounded. However strong, though, the notion of a topological group satisfying $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ is not trivial.

Theorem 18. *For each cardinal κ with $\mathsf{cf}(\kappa) > \aleph_0$, it is consistent that $\mathfrak{c} = \kappa$ and there exists a topological subgroup of $\mathbb{R}$ of size $\mathfrak{c}$, which satisfies $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$.*

Proof. This is really a theorem of Miller: Let M be a countable standard model of ZFC satisfying $\mathfrak{c} = \kappa$. In [14] it is proved that there exists a ccc poset $\mathbb{P}$ in M of size continuum (so that forcing with $\mathbb{P}$ does not change the size of the continuum) such that the old reals $M \cap \mathbb{R}$ satisfy $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ in $V^{\mathbb{P}}$.

But observe that, as the operations of addition and substraction in $\mathbb{R}$ are absolute, $G = M \cap \mathbb{R}$ is a group in $V^{\mathbb{P}}$. $\qquad\square$

Problem 19. *Does the Continuum Hypothesis imply the existence of a separable metrizable group of size $\mathfrak{c}$ which satisfies $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$?*

We have some approximate results. For a sequence $\{X_n\}_{n\in\mathbb{N}}$ of subsets of X, define $\liminf X_n = \bigcup_m \bigcap_{n \geq m} X_n$. For a family $\mathcal{F}$ of subsets of X, $L(\mathcal{F})$ denotes its closure under the operation $\liminf$. According to [8], X is a δ-*set* if for each ω-cover $\mathcal{U}$ of X, $X \in L(\mathcal{U})$. It is easy to see that the δ-property is preserved under taking countable *increasing* unions. Clearly $\mathsf{S}_1(\Omega, \Gamma)$ implies the δ-property. The converse is still an open problem [19]. If a δ-set is a group, we will call it a δ-*group*.

Let $\mathbb{Z}_2$ denote the usual group $\{0, 1\}$ with modulo 2 addition.

Theorem 20. *Assume that $\mathfrak{p} = \mathfrak{c}$. Then there exists a subgroup G of $^{\mathbb{N}}\mathbb{Z}_2$ such that for each k G^k is a countable increasing union of sets satisfying $\mathsf{S}_1(\Omega, \Gamma)$. In particular, all finite powers of G are δ-groups.*

Proof. As $\mathfrak{p} = \mathfrak{c}$, there exists a subset X of $^{\mathbb{N}}\mathbb{Z}_2$ of size continuum which satisfies $\mathsf{S}_1(\Omega, \Gamma)$ [7]. We may assume that $0 \in X$. Consequently,

$$G := \langle X \rangle = \bigcup_{n \in \mathbb{N}} \{x_1 + \cdots + x_n : x_1, \ldots, x_n \in X\}$$

is a countable increasing union of continuous images of powers of X. But the property $\mathsf{S}_1(\Omega, \Gamma)$ is closed under taking finite powers and continuous images [11]. Observe that each finite power of G is the countable increasing union of the same power of the original sets. $\qquad\square$

Assuming the Continuum Hypothesis, there exists a set of reals X of size $\mathfrak{c}$ satisfying $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ (e.g., [14]). It is an open problem whether $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ is provably preserved under taking finite powers [19]. If it is, then we have a positive answer to Problem 19.

Theorem 21. *Assume that $X \subseteq {}^{\mathbb{N}}\mathbb{Z}_2$ satisfies $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ in all finite powers. Then $G = \langle X \rangle$ satisfies $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ in all finite powers.*

Proof. By the above arguments, it suffices to prove the following.

Lemma 22. $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ *is preserved under taking countable increasing unions.*

To prove the lemma, assume that $X = \bigcup_n X_n$ is an increasing union, and observe that $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$ implies $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, which in turn implies that all Borel images of each X_k in $^\mathbb{N}\mathbb{N}$ are bounded [17].

Assume that $\mathcal{U}_n = \{U_m^n : m \in \mathbb{N}\}$, $n \in \mathbb{N}$, are countable Borel ω-covers of X. By $\mathsf{S}_1(\mathcal{B}_\Omega, \mathcal{B}_\Gamma)$, we may assume that each $\mathcal{U}_n$ is a γ-cover of X_n. For each k, define a function Ψ_k from X_k to $^\mathbb{N}\mathbb{N}$ so that for each x and n,

$$\Psi_k(x)(n) = \min\{i : (\forall m \geq i)\ x \in U_m^n\}.$$

Ψ_k is a Borel map, thus $\Psi_k[X]$ is bounded. Consequently, $\bigcup_k \Psi_k[X]$ is bounded, say by the sequence m_n. Then $\{U_{m_n}^n\}_{n \in \mathbb{N}}$ is a γ-cover of X, as required. $\qquad\square$

We conclude the paper with a group satisfying $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ in all finite powers. Let $\mathcal{N}$ denote the collection of null (Lebesgue measure zero) sets of reals. $\mathsf{cov}(\mathcal{N})$ is the minimal size of a cover of $\mathbb{R}$ by null sets, and $\mathsf{cof}(\mathcal{N})$ is the minimal size of a cofinal family in $\mathcal{N}$ with respect to inclusion. Let κ be an uncountable cardinal. $S \subseteq \mathbb{R}$ is a κ-Sierpiński set if $|S| \geq \kappa$ and for each null set N, $|S \cap N| < \kappa$. $\mathfrak{b}$-Sierpiński sets satisfy $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, but powers of κ-Sierpiński sets are never κ-Sierpiński sets.

Theorem 23. *Assume that* $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N}) = \mathfrak{b}$. *Then there exists a (non-σ-compact) group* $G \subseteq \mathbb{R}$ *of size* $\mathfrak{b}$, *such that all finite powers of* G *satisfy* $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. *Consequently, all finite powers of* G *also satisfy* $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$.

Proof. The proof is similar to that of Theorem 7.2 of [6].

Lemma 24. *Assume that* $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N})$. *Then there exists a* $\mathsf{cov}(\mathcal{N})$-*Sierpiński set* S *such that for each* k *and each null set* N *in* $\mathbb{R}^k$, $S^k \cap N$ *is contained in a union of less than* $\mathsf{cov}(\mathcal{N})$ *many continuous images of* S^{k-1}.

Proof. Let $\kappa = \mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N})$. Let $\{N_\alpha^{(k)} : \alpha < \kappa\}$ be a cofinal family of null sets in $\mathbb{R}^k$, $k \in \mathbb{N}$.

For $J \subseteq \mathbb{R}^k$, $x \in \mathbb{R}$, and $i < k$, define $J_{(x,i)} = \{v_1 \frown v_2 : \ell(v_1) = i,\ v_1 \frown x \frown v_2 \in J\}$. By the Fubini Theorem, for each null set $N \subseteq \mathbb{R}^k$ and $i < k$,

$$\tilde{N} = \{x : (\exists i < k)\ N_{(x,i)} \text{ is not null in } \mathbb{R}^{k-1}\} \in \mathcal{N}.$$

We make an inductive construction on $\alpha < \kappa$ of elements $x_\alpha \in \mathbb{R}$ with auxiliary collections $\mathcal{F}_\alpha$ of null sets, as follows. For $\alpha < \kappa$ let $\mathcal{P}_\alpha = \{N_\alpha^{(k)} : k \in \mathbb{N}\}$. At step α do the following:

(1) Choose $x_\alpha \notin \bigcup_{\beta < \alpha} \left(\bigcup_{N \in (\mathcal{P}_\beta \cup \mathcal{F}_\beta) \setminus P(\mathbb{R})} \tilde{N} \cup \bigcup_{N \in (\mathcal{P}_\beta \cup \mathcal{F}_\beta) \cap P(\mathbb{R})} N \right)$.

(2) Set $\mathcal{F}_\alpha = \{N_{(x_\alpha, i)} : \beta < \alpha,\ N \in (\mathcal{P}_\beta \cup \mathcal{F}_\beta) \setminus P(\mathbb{R}),\ i \in \mathbb{N}\}$.

This is possible because x_α is required to avoid membership in a union of less than $\mathsf{cov}(\mathcal{N})$ many null sets.

Take $S = \{x_\alpha : \alpha < \kappa\}$. Then S is a κ-Sierpiński set. Fix k. For each null $N \subseteq \mathbb{R}^k$, there exists $\beta < \kappa$ with $N \subseteq N_\beta^{(k)}$. Whenever $\beta < \alpha_0 < \cdots < \alpha_{k-1}$, and π is a permutation on $\{0, \ldots, k-1\}$, $(N_\beta^{(k)})_{(x_{\alpha_0}, \pi^{-1}(0))} \in \mathcal{F}_{\alpha_0}$, thus $(N_\beta^{(k)})_{(x_{\alpha_0}, \pi^{-1}(0)), (x_{\alpha_1}, \pi^{-1}(1))} \in \mathcal{F}_{\alpha_1}, \ldots, (N_\beta^{(k)})_{(x_{\alpha_0}, \pi^{-1}(0)), \ldots, (x_{\alpha_{k-2}}, \pi^{-1}(k-2))} \in \mathcal{F}_{\alpha_{k-2}}$, thus $x_{\alpha_{k-1}} \notin (N_\beta^{(k)})_{(x_{\alpha_0}, \pi^{-1}(0)), \ldots, (x_{\alpha_{k-2}}, \pi^{-1}(k-2))}$, that is, $(x_{\alpha_{\pi(0)}}, \ldots, x_{\alpha_{\pi(k-1)}}) \notin N_\beta^{(k)}$.

Consequently, $S^k \cap N$ is contained in the union of all sets of the form $S^i \times \{x_\xi : \xi \leq \beta\} \times S^{k-i-1}$ $(i < k)$—a union of $|\beta| < \kappa$ copies of S^{k-1}—and $\{v \in S^k : v_i = v_j\}$ $(i < j < k)$, which are continuous images of S^{k-1}. $\hfill\square$

Now assume that $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N}) = \mathfrak{b}$, and let S be a $\mathfrak{b}$-Sierpiński set as in Lemma 24. We will show by induction that for each k, S^k satisfies $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. By [17] it is enough to show that for each null $N \subseteq \mathbb{R}^k$, $S^k \cap N$ satisfies $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. By Lemma 24 and the induction hypothesis, $S^k \cap N$ is contained in a union of less than $\mathfrak{b}$ many sets satisfying $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$. In [4, full version] it is shown that $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is preserved under taking unions of size less than $\mathfrak{b}$, and in [5] it is shown that $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is preserved under taking subsets. This proves the assertion.

So all finite powers of S satisfy $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$, and $G = \langle S \rangle$ works, since as we mentioned before, $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ is preserved under taking countable unions. Finally, by [17] $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B})$ in all finite powers implies $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$. $\hfill\square$

It seems that the following was not known before.

Corollary 25. *Assume that* $\mathsf{cov}(\mathcal{N}) = \mathsf{cof}(\mathcal{N}) = \mathfrak{b}$. *Then there exists a set of reals satisfying* $\mathsf{S}_1(\mathcal{B}_\Gamma, \mathcal{B}_\Gamma)$ *and* $\mathsf{S}_{fin}(\mathcal{B}_\Omega, \mathcal{B}_\Omega)$, *but not* $\mathsf{S}_1(\mathcal{O}, \mathcal{O})$.

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