Topology and its Applications 192 (2015) 198–207

Contents lists available at ScienceDirect

Topology and its Applications

www.elsevier.com/locate/topol

Combinatorial aspects of selective star covering properties in Ψ-spaces

Boaz Tsaban [a,b,*]

[a] *Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel*
[b] *Faculty of Mathematics and Computer Science, Weizmann Institute of Science, Rehovot 7610001, Israel*

ARTICLE INFO

Article history:
Received 28 May 2014
Accepted 15 December 2014
Available online 1 June 2015

MSC:
54D20
54A35
03E17

Keywords:
Isbell–Mrówka space
Ψ-space
Star-Menger
Star-Hurewicz

ABSTRACT

Which Isbell–Mrówka spaces (Ψ-spaces) satisfy the star version of Menger's and Hurewicz's covering properties? Following Bonanzinga and Matveev, this question is considered here from a combinatorial point of view. An example of a Ψ-space that is (strongly) star-Menger but not star-Hurewicz is obtained. The PCF-theory function $\kappa \mapsto \mathrm{cof}([\kappa]^{\aleph_0})$ is a key tool. Using the method of forcing, a complete answer to a question of Bonanzinga and Matveev is provided.

The results also apply to the mentioned covering properties in the realm of Pixley–Roy spaces, to the extent of spaces with these properties, and to the character of free abelian topological groups over hemicompact k spaces.

1. Introduction

The Isbell–Mrówka Ψ-spaces [11,16] are classic examples in the realm of topological covering properties. A family $\mathcal{A} \subseteq P(\mathbb{N})$ is *almost disjoint* if every element of $\mathcal{A}$ is infinite, and the sets $A \cap B$ are finite for all distinct elements $A, B \in \mathcal{A}$. For an almost disjoint family $\mathcal{A}$, let $\Psi(\mathcal{A}) := \mathcal{A} \cup \mathbb{N}$. A topology on $\Psi(\mathcal{A})$ is defined as follows. The natural numbers are isolated, and for each element $A \in \mathcal{A}$ and each finite set $F \subseteq \mathbb{N}$, the set $\{A\} \cup (A \setminus F)$ is a basic open neighborhood of A. Spaces constructed in this manner are called *Ψ-spaces*.

For a set X, a subset A of X and a family $\mathcal{U}$ of subsets of X, let $\mathrm{star}(A,\mathcal{U}) := \bigcup\{U \in \mathcal{U} : A \cap U \neq \emptyset\}$. A topological space X is *star-Lindelöf* [5] if every open cover $\mathcal{U}$ of X has a countable subset $\mathcal{V}$ such that $X = \mathrm{star}(\bigcup\mathcal{V},\mathcal{U})$. It is *strongly star-Lindelöf* [5] if, for each open cover $\mathcal{U}$ of X, there is a countable set

* Correspondence to: Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel
E-mail address: tsaban@math.biu.ac.il.
URL: http://math.biu.ac.il/~tsaban.

http://dx.doi.org/10.1016/j.topol.2015.05.082
0166-8641/

$C \subseteq X$ such that $X = \mathrm{star}(C, \mathcal{U})$. It is easy to see that uncountable Ψ-spaces are not Lindelöf. Being separable, though, all Ψ-spaces are strongly star-Lindelöf.

Menger's property is the following selective version of Lindelöf's property: For every sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that the family $\{\bigcup \mathcal{F}_1, \bigcup \mathcal{F}_2, \dots\}$ covers X.

A topological space X is *star-Menger* (respectively, *strongly star-Menger*) [13] if for every sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ (respectively, $F_1, F_2, \dots \subseteq X$) such that the family $\{\mathrm{star}(\bigcup \mathcal{F}_1, \mathcal{U}_1), \mathrm{star}(\bigcup \mathcal{F}_2, \mathcal{U}_2), \dots\}$ (respectively, $\{\mathrm{star}(F_1, \mathcal{U}_1), \mathrm{star}(F_2, \mathcal{U}_2), \dots\}$) covers X.

A topological space X is a *Hurewicz* (respectively: *star-Hurewicz*; *strongly star-Hurewicz*) *space* [3] if, in the corresponding definitions in the previous paragraph, we request that every point of X is in the set $\bigcup \mathcal{F}_n$ (respectively: $\mathrm{star}(\bigcup \mathcal{F}_n, \mathcal{U}_n)$; $\mathrm{star}(F_n, \mathcal{U}_n)$) for all but finitely many n.

The implications among the mentioned covering properties are as follows.

$$\begin{tikzcd}
\text{Lindelöf} \arrow[r] & \text{strongly star-Lindelöf} \arrow[r] & \text{star-Lindelöf} \\
\text{Menger} \arrow[u] \arrow[r] & \text{strongly star-Menger} \arrow[u] \arrow[r] & \text{star-Menger} \arrow[u] \\
\text{Hurewicz} \arrow[u] \arrow[r] & \text{strongly star-Hurewicz} \arrow[u] \arrow[r] & \text{star-Hurewicz} \arrow[u]
\end{tikzcd}$$

A survey of these properties and their connections to other notions is available in [14].

Background on the combinatorial cardinals of the continuum used in this paper, including the unbounding number $\mathfrak{b}$ and the dominating number $\mathfrak{d}$, is available in [4,2]. Whether a Ψ-space is strongly star-Menger—or strongly star-Hurewicz—depends only on the cardinality of the space.

Theorem 1.1. *(Bonanzinga–Matveev [7]) Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family.*

(1) *The space $\Psi(\mathcal{A})$ is strongly star-Menger if and only if $|\mathcal{A}| < \mathfrak{d}$.*
(2) *The space $\Psi(\mathcal{A})$ is strongly star-Hurewicz if and only if $|\mathcal{A}| < \mathfrak{b}$.*

The question of when a Ψ-space $\Psi(\mathcal{A})$ is star-Menger—or star-Hurewicz—is more elusive. Combinatorial characterizations in terms of the family $\mathcal{A}$ are provided in Section 2, but some of the most basic problems remain, in general, open. Some of these problems are reviewed in Section 4.

Let P be a partially ordered set. A subset C of P is *cofinal* if for each element $a \in P$ there is an element $c \in C$ such that $a \leq c$. The *cofinality* of P, denoted $\mathrm{cof}(P)$, is the minimal cardinality of a cofinal subset of P. The number $\mathrm{cof}(P)$ may, in general, be a singular cardinal number. For a set X, let $\mathrm{Fin}(X)$ be the family of all finite subsets of X. In this paper, families of sets are always partially ordered by the relation $\subseteq$. The set $\mathrm{Fin}(X)^{\mathbb{N}}$ of all functions $f : \mathbb{N} \to \mathrm{Fin}(X)$ is partially ordered coordinate-wise: $f \leq g$ if $f(n) \subseteq g(n)$ for all n. The cardinal $\mathrm{cof}(\mathrm{Fin}(X)^{\mathbb{N}})$ depends only on $|X|$. For an infinite cardinal κ, the cardinal $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}})$ will later be expressed in simpler terms. In particular, it is known that the cardinality $\mathfrak{c}$ of the continuum satisfies $\mathrm{cof}(\mathrm{Fin}(\mathfrak{c})^{\mathbb{N}}) = \mathfrak{c}$.

Theorem 1.2. *(Bonanzinga–Matveev [7]) Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family of cardinality κ. If $\mathrm{cof}(\mathrm{Fin}(\kappa)^{\mathbb{N}}) = \kappa$, then the space $\Psi(\mathcal{A})$ is not star-Menger.*

A simple proof of Theorem 1.2 is provided in Section 2. Section 2 also includes a similar theorem for star-Hurewicz Ψ-spaces (Theorem 2.4). Theorems 1.1(1) and 2.4 are used in Example 2.5 to obtain a consistent example of a (strongly) star-Menger Ψ-space that is not star-Hurewicz.

The existence of a star-Menger Ψ-space that is not star-Hurewicz violates the Continuum Hypothesis, and thus cannot be constructed in ZFC alone. Indeed, Ψ-spaces have cardinality at most $\mathfrak{c}$. Since $\operatorname{cof}(\operatorname{Fin}(\mathfrak{c})^{\mathbb{N}}) = \mathfrak{c}$, every star-Menger Ψ-space has cardinality smaller than $\mathfrak{c}$. By Theorem 1.1(2), we have the following corollary.

Corollary 1.3. *If $\mathfrak{b} = \mathfrak{c}$, then every star-Menger Ψ-space is (strongly) star-Hurewicz.* □

Remark 1.4. If we do not insist on Ψ-spaces then there is, provably in ZFC, a very nice (strongly) star-Menger space that is not star-Hurewicz: For paracompact spaces, each of the mentioned covering properties coincides with its star- and strongly star- versions. Chaber and Pol proved that there are Menger subsets of the Cantor space that are not Hurewicz (cf. [18]).

The question whether $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \kappa$ for a cardinal number κ appears in a number of additional, related and seemingly unrelated, topological contexts. The following theorem follows from Sakai's Theorem 2.1 in [17], since being closed discrete is a hereditary property.

Theorem 1.5. *(Sakai) Let D be a closed discrete subspace of a regular strongly star-Menger space. Then the cardinality of D is smaller than the minimal fixed point of the function $\kappa \mapsto \operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}})$.*

Let X be a topological space. The *Pixley–Roy space* $\operatorname{PR}(X)$ is the space of all nonempty finite subsets of X, with the topology determined by the basic open sets

$$[F, U] := \{H \in \operatorname{PR}(X) : F \subseteq H \subseteq U\},$$

$F \in \operatorname{PR}(X)$ and U open in X.

Theorem 1.6. *(Sakai [17]) Let X be an infinite regular topological space of cardinality κ. If $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \kappa$, then the space $\operatorname{PR}(X)$ is not star-Menger.*

The cardinals $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}})$ also show up in a study of the character of topological groups.

Theorem 1.7. *([6]) Let X be a nondiscrete hemicompact k space. Let κ be the supremum of the weights of compact subsets of X. Then the character of the free abelian topological group $A(X)$ is $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}})$.*

A similar result is proved in [6] for general abelian non-locally compact hemicompact k groups. A number of estimations of $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}})$ for infinite cardinals κ are provided there. The key to these is the following reduction. For an infinite cardinal number κ, let $[\kappa]^{\aleph_0}$ be the family of all countably infinite subsets of κ.

Proposition 1.8. *([6]) Let κ be an infinite cardinal number. Then $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}})$ is the maximum of the cardinals $\mathfrak{d}$ and $\operatorname{cof}([\kappa]^{\aleph_0})$.*

Thus, the Bonanzinga–Matveev Theorem 1.2 can be reformulated as follows. (Recall that the space $\Psi(\mathcal{A})$ is strongly star-Menger if $|\mathcal{A}| < \mathfrak{d}$.)

Theorem 1.9. *Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family of cardinality $\kappa \geq \mathfrak{d}$. If $\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa$, then the space $\Psi(\mathcal{A})$ is not star-Menger.*

The estimation of the cardinal $\operatorname{cof}([\kappa]^{\aleph_0})$ in terms of the cardinal κ is a central goal in Shelah's *PCF theory*, the theory of possible cofinalities. In contrast to cardinal exponentiation, the function $\kappa \mapsto \operatorname{cof}([\kappa]^{\aleph_0})$ is tame. For example, if there are no large cardinals in the Dodd–Jensen core model, then $\operatorname{cof}([\kappa]^{\aleph_0})$ is simply κ if κ has uncountable cofinality, and κ^+ (the successor of κ) otherwise [8]. Moreover, without any special hypotheses, the cardinal $\operatorname{cof}([\kappa]^{\aleph_0})$ can be estimated, and in many cases computed exactly. Some examples follow (for proofs and references, see [6, Section 8]).

For uncountable cardinals κ of countable cofinality, a variation of König's Lemma implies that $\operatorname{cof}([\kappa]^{\aleph_0}) > \kappa$. Throughout, *Shelah's Strong Hypothesis (SSH)* is the assertion that $\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa^+$ for all uncountable cardinals κ of countable cofinality. Clearly, the Generalized Continuum Hypothesis implies SSH, but the latter axiom is much weaker, being a consequence of the absence of large cardinals.

Theorem 1.10. *(Folklore) The following cardinals are fixed points of the function $\kappa \mapsto \operatorname{cof}([\kappa]^{\aleph_0})$:*

(1) *The cardinals κ with $\kappa^{\aleph_0} = \kappa$.*
(2) *$\aleph_n$, for natural numbers $n \geq 1$.*
(3) *The cardinals $\aleph_\kappa$, for κ a singular cardinal of uncountable cofinality that is smaller than the first fixed point of the $\aleph$ function.*
(4) *Assuming SSH, all cardinals of uncountable cofinality.*

Moreover, successors of fixed points of this function are also fixed points.

For example, for $n = 1, 2, \dots$, the cardinal $\aleph_{\aleph_{\omega_n}}$ and its successors are all fixed points of the function $\kappa \mapsto \operatorname{cof}([\kappa]^{\aleph_0})$.

Corollary 1.11. *Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family of cardinality at least $\mathfrak{d}$.*

(1) *For each cardinal κ smaller than the first fixed point of the $\aleph$ function, with $\aleph_0 < \operatorname{cof}(\kappa) < \kappa$, if $|\mathcal{A}| = \aleph_\alpha$ for some ordinal α with $\kappa \leq \alpha < \kappa + \omega$, then the space $\Psi(\mathcal{A})$ is not star-Menger.*
(2) *Assume SSH. If the cardinal $|\mathcal{A}|$ has uncountable cofinality, then the space $\Psi(\mathcal{A})$ is not star-Menger.* $\square$

The cardinality of Ψ-spaces is at most $\mathfrak{c}$. Knowing that $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d} \cdot \kappa$ for the cardinals $\aleph_n$ (for $n \in \mathbb{N}$) and for the cardinal $\mathfrak{c}$, the following problem is natural.

Problem 1.12. (Bonanzinga–Matveev [7]) Is $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}\cdot\kappa$ for each infinite cardinal $\kappa \leq \mathfrak{c}$? In particular, is $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}$ for each infinite cardinal $\kappa \leq \mathfrak{d}$?

This problem is solved in Section 3.

2. Combinatorial characterizations and a consequence

The following theorem provides a combinatorial characterization of star-Menger Ψ-spaces.

Theorem 2.1. *Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family. The following assertions are equivalent:*

(1) *The Isbell–Mrówka space $\Psi(\mathcal{A})$ is star-Menger.*
(2) *For each function $A \mapsto f_A$ from $\mathcal{A}$ to $\mathbb{N}^{\mathbb{N}}$, there are finite sets $\mathcal{F}_1, \mathcal{F}_2, \dots \subseteq \mathcal{A}$ such that, for each $A \in \mathcal{A}$, there is n with $(A \setminus f_A(n)) \cap \bigcup_{B\in\mathcal{F}_n}(B \setminus f_B(n)) \neq \emptyset$.*

Proof. (2) $\Rightarrow$ (1): Since the subspace $\mathbb{N}$ of $\Psi(\mathcal{A})$ is countable, it suffices in the definition of the star-Menger property to cover $\mathcal{A}$. Let $\mathcal{U}_n$, for $n \in \mathbb{N}$, be open covers of $\Psi(\mathcal{A})$. By moving to a finer open cover, we may assume that for each $A \in \mathcal{A}$ and each n, there is a natural number $f_A(n)$ such that $\{A\} \cup (A \setminus f_A(n)) \in \mathcal{U}_n$.

Let $\mathcal{F}_1, \mathcal{F}_2, \dots \subseteq \mathcal{A}$ be finite sets as in (2). For each n, the set

$$\{\{B\} \cup (B \setminus f_B(n)) : B \in \mathcal{F}_n\}$$

is a finite subset of $\mathcal{U}_n$. Let $A \in \mathcal{A}$. Pick n as in (2). Then

$$A \in \{A\} \cup (A \setminus f_A(n)) \subseteq \operatorname{star}\big(\textstyle\bigcup_{B \in \mathcal{F}_n} (\{B\} \cup (B \setminus f_B(n))), \mathcal{U}_n\big).$$

(1) ⇒ (2): For each n, let

$$\mathcal{U}_n := \{\{A\} \cup (A \setminus f_A(n)) : A \in \mathcal{A}\} \cup \{\{m\} : m \in \mathbb{N}\}.$$

Since the space $\Psi(\mathcal{A})$ is star-Menger, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1, \mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ such that $\Psi(\mathcal{A}) = \bigcup_n \operatorname{star}(\bigcup \mathcal{F}_n, \mathcal{U}_n)$. For each n and each $\{m\} \in \mathcal{F}_n$, pick if possible an element $B \in \mathcal{A}$ such that $m \in B \setminus f_B(n)$, and substitute $\{B\} \cup (B \setminus f_B(n))$ for $\{m\}$ in $\mathcal{F}_n$. If there is no such B, just remove $\{m\}$ from $\mathcal{F}_n$ (in this case, $\operatorname{star}(\{m\}, \mathcal{U}_n) = \{m\}$). Then $\mathcal{A} \subseteq \bigcup_n \operatorname{star}(\bigcup \mathcal{F}_n, \mathcal{U}_n)$. The assertion in (2) then follows from the definitions. $\square$

We obtain the following simple proof of Theorem 1.2. The main simplification over the proof in [7] is that we avoid the necessity to use two types of cofinal sets simultaneously.

Proof of Theorem 1.2. We establish the negation of the characterization in Theorem 2.1.

Enumerate $\mathcal{A} := \{A_\alpha : \alpha < \kappa\}$, and let $\{F_\alpha : \alpha < \kappa\}$ be a cofinal subset of $\operatorname{Fin}(\kappa)^{\mathbb{N}}$. We may assume that $\alpha \notin F_\alpha(n)$ for all n. Indeed, the family $\{F'_\alpha : \alpha < \kappa\}$, defined by $F'_\alpha(n) := F_\alpha(n) \setminus \{\alpha\}$ for all n, is cofinal in $\operatorname{Fin}(\kappa)^{\mathbb{N}}$: Let $F \in \operatorname{Fin}(\kappa)^{\mathbb{N}}$, and set $I := \{\alpha < \kappa : F \leq F_\alpha\}$. For each ordinal $\beta < \kappa$, there is $\alpha < \kappa$ such that $F(n) \cup \{\beta\} \subseteq F_\alpha(n)$ for all n. Thus, $\bigcup_{\alpha \in I} \bigcup_n F_\alpha(n) = \kappa$, and therefore the set I is uncountable. Pick an ordinal $\alpha \in I \setminus \bigcup_n F(n)$. Then $F(n) \subseteq F_\alpha(n) \setminus \{\alpha\}$ for all n.

For each $\alpha < \kappa$ and each n, let

$$f_\alpha(n) := 1 + \max \bigcup_{\beta \in F_\alpha(n)} A_\alpha \cap A_\beta.$$

Let $\mathcal{F}_1, \mathcal{F}_2, \dots \subseteq \mathcal{A}$ be finite sets. For each n, let $H_n := \{\alpha < \kappa : A_\alpha \in \mathcal{F}_n\}$. Take α such that $H_n \subseteq F_\alpha(n)$ for all n. Then, for each n, we have that $\max(A_\alpha \cap \bigcup_{\beta \in H_n} A_\beta) < f_\alpha(n)$, and thus

$$(A_\alpha \setminus f_\alpha(n)) \cap \bigcup_{\beta \in H_n} (A_\beta \setminus f_\beta(n)) \subseteq (A_\alpha \setminus f_\alpha(n)) \cap \bigcup_{\beta \in H_n} A_\beta = \emptyset. \qquad \square$$

The following theorem provides a combinatorial characterization of star-Hurewicz Ψ-spaces. Its proof, which is similar to that of Theorem 2.1, is omitted.

Theorem 2.2. *Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family. The following assertions are equivalent:*

(1) *The Isbell–Mrówka space $\Psi(\mathcal{A})$ is star-Hurewicz.*
(2) *For each function $A \mapsto f_A$ from $\mathcal{A}$ to $\mathbb{N}^{\mathbb{N}}$, there are finite sets $\mathcal{F}_1, \mathcal{F}_2, \dots \subseteq \mathcal{A}$ such that, for each $A \in \mathcal{A}$, $(A \setminus f_A(n)) \cap \bigcup_{B \in \mathcal{F}_n} (B \setminus f_B(n)) \neq \emptyset$ for all but finitely many n.* $\square$

Proposition 2.3. *Let κ be an infinite cardinal. The following cardinal numbers are equal:*

(1) *The minimal cardinality of a family $\mathcal{F} \subseteq \mathrm{Fin}(\kappa)^{\mathbb{N}}$ such that for each $g \in \mathrm{Fin}(\kappa)^{\mathbb{N}}$ there is $f \in \mathcal{F}$ with $g(n) \subseteq f(n)$ for infinitely many n.*
(2) *The maximum of the cardinals $\mathfrak{b}$ and $\mathrm{cof}([\kappa]^{\aleph_0})$.*

Proof. (2) $\leq$ (1): Let $\mathcal{F}$ be as in (1).

For each $f \in \mathcal{F}$, define a function $f' \in \mathbb{N}^{\mathbb{N}}$ by

$$f'(n) := 1 + \max(f(n) \cap \mathbb{N}).$$

For each function $g \in \mathbb{N}^{\mathbb{N}}$, there is $f \in \mathcal{F}$ such that $\{1, \dots, g(n)\} \subseteq f(n)$, and thus $g(n) \leq f'(n)$, for infinitely many n. Thus, the family $\{f' : f \in \mathcal{F}\}$ is unbounded. This shows that $\mathfrak{b} \leq |\mathcal{F}|$.

For each set $A \in [\kappa]^{\aleph_0}$, pick a function $g \in \mathrm{Fin}(\kappa)^{\mathbb{N}}$ such that $g(n) \subseteq g(n+1)$ for all n, and $A \subseteq \bigcup_n g(n)$. Pick $f \in \mathcal{F}$ such that $g(n) \subseteq f(n)$ for infinitely many n. Then, since $g(n) \subseteq g(n+1)$ for all n, $\bigcup_n g(n) \subseteq \bigcup_n f(n)$. Thus, the family $\{\bigcup_n f(n) : f \in \mathcal{F}\}$ is cofinal in $[\kappa]^{\aleph_0}$. It follows that $\mathrm{cof}([\kappa]^{\aleph_0}) \leq |\mathcal{F}|$.

(1) $\leq$ (2): Let $\mathcal{G}$ be an unbounded family in $\mathbb{N}^{\mathbb{N}}$, and $\mathcal{H}$ be a cofinal family in $[\kappa]^{\aleph_0}$. For each set $A \in \mathcal{H}$, fix a function $f_A \in \mathrm{Fin}(\kappa)^{\mathbb{N}}$ such that $f_A(n) \subseteq f_A(n+1)$ for all n, and $A \subseteq \bigcup_n f_A(n)$.

Let $h \in \mathrm{Fin}(\kappa)^{\mathbb{N}}$. Pick $A \in \mathcal{H}$ with $\bigcup_n h(n) \subseteq A$. Pick $g \in \mathcal{G}$ such that

$$\min\{m : h(n) \subseteq f_A(m)\} \leq g(n)$$

for infinitely many n. Then $h(n) \subseteq f_A(g(n))$ for infinitely many n. Take $\mathcal{F} := \{f_A \circ g : g \in \mathcal{G}, A \in \mathcal{H}\}$. Then $|\mathcal{F}| \leq \mathfrak{b} \cdot \mathrm{cof}([\kappa]^{\aleph_0})$. $\square$

We obtain the following analogue of Theorem 1.9. (Recall that Ψ-spaces of cardinality smaller than $\mathfrak{b}$ are strongly star-Hurewicz.)

Theorem 2.4. *Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family of cardinality $\kappa \geq \mathfrak{b}$. If $\mathrm{cof}([\kappa]^{\aleph_0}) = \kappa$, then the space $\Psi(\mathcal{A})$ is not star-Hurewicz.*

Proof. The proof is almost identical to that of Theorem 1.2, using Proposition 2.3 and Theorem 2.2. The necessary changes are as follows. Here, we let $\{F_\alpha : \alpha < \kappa\} \subseteq \mathrm{Fin}(\kappa)^{\mathbb{N}}$ be a family as in Proposition 2.3(1). For the last step of the proof, we take α such that $H_n \subseteq F_\alpha(n)$ for infinitely many n, and restrict attention to these n. $\square$

Example 2.5. Assume that $\mathfrak{b} = \aleph_1 < \mathfrak{d}$. Then there is a strongly star-Menger Ψ-space that is not star-Hurewicz.

Proof. Since there are almost disjoint sets of cardinality continuum, there are ones of any smaller cardinality, too. Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family of cardinality $\aleph_1$. By Theorem 1.1, the space $\Psi(\mathcal{A})$ is strongly star-Menger. By Theorem 1.10(2) and Theorem 2.4, this space is not star-Hurewicz. $\square$

Corollary 2.6. *(SSH) The following assertions are equivalent:*

(1) *There is a strongly star-Menger Ψ-space that is not star-Hurewicz.*
(2) $\mathfrak{b} < \mathfrak{d}$.

Proof. (1) ⇒ (2): Let $\Psi(\mathcal{A})$ exemplify (1). By Theorem 1.1, $|\mathcal{A}| < \mathfrak{d}$. If $\mathfrak{b} = \mathfrak{d}$ then, by the same theorem, the space $\Psi(\mathcal{A})$ is (strongly) star-Hurewicz; a contradiction.

(2) ⇒ (1): Take a Ψ-space of cardinality $\mathfrak{b}$. By Theorem 1.1, the space $\Psi(\mathcal{A})$ is strongly star-Menger. By Theorem 1.10(4), since $\mathfrak{b}$ is a regular cardinal, $\operatorname{cof}([\mathfrak{b}]^{\aleph_0}) = \mathfrak{b}$. Apply Theorem 2.4. □

3. A solution of the Bonanzinga–Matveev problem

Problem 1.12 asks whether $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}\cdot\kappa$ for each infinite cardinal $\kappa \leq \mathfrak{c}$, and, in particular, whether $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}$ for each infinite cardinal $\kappa \leq \mathfrak{d}$.

Clearly, the Continuum Hypothesis implies a positive answer to Problem 1.12, and Problem 1.12 actually asks whether the assertions are provable without special set theoretic hypotheses. We first point out a negative answer to the first part of this problem.

Proposition 3.1. *Let $\aleph_\alpha := \mathfrak{d}$. If $\aleph_{\alpha+\omega} < \mathfrak{c}$, then there is a cardinal $\kappa < \mathfrak{c}$ such that $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) > \mathfrak{d}\cdot\kappa$.*

Proof. Take $\kappa := \aleph_{\alpha+\omega}$. Since $\mathfrak{d} \leq \kappa \leq \operatorname{cof}([\kappa]^{\aleph_0})$, we have by Theorem 1.10 that $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \operatorname{cof}([\kappa]^{\aleph_0})$. By König's Lemma, we have that $\operatorname{cof}([\kappa]^{\aleph_0}) > \kappa = \mathfrak{d}\cdot\kappa$. □

We use some facts from the theory of forcing. A general introduction is available in Kunen's book [15], whose notation we follow. Some more details that are relevant for us here are available in Bartoszyński and Judah's book [1], and in Blass' chapter [2].

Fix a successor ordinal $\beta > \omega$. Adding $\aleph_\beta$ random reals to a model of the Continuum Hypothesis, we obtain a model of $\mathfrak{d} = \aleph_1$ and $\mathfrak{c} = \aleph_\beta$. Such a model satisfies the condition in Proposition 3.1.

SSH implies a positive answer to the second part of the Bonanzinga–Matveev problem, and a conditional solution to its first part.

Theorem 3.2. *(SSH)*

(1) *For each infinite cardinal $\kappa \leq \mathfrak{d}$, we have that $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}$.*
(2) *$\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}\cdot\kappa$ for all infinite cardinals $\kappa \leq \mathfrak{c}$ if, and only if, there is $n \geq 0$ such that $\mathfrak{c} = \mathfrak{d}^{+n}$, the n-th successor of $\mathfrak{d}$.*

Proof. We use Theorem 1.10.

(1) If $\operatorname{cof}(\kappa) > \aleph_0$, then $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}\cdot\kappa = \mathfrak{d}$. Otherwise, as $\operatorname{cof}(\mathfrak{d}) \geq \mathfrak{b} > \aleph_0$, we have that $\kappa < \mathfrak{d}$, and $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}\cdot\kappa^+ = \mathfrak{d}$.

(2) If there is such n, then each κ with $\mathfrak{d} \leq \kappa \leq \mathfrak{c}$ has uncountable cofinality, and by SSH we have that $\operatorname{cof}(\operatorname{Fin}(\kappa)^{\mathbb{N}}) = \mathfrak{d}\cdot\kappa$. Otherwise, Proposition 3.1 applies. □

Thus, the answer to the first part of Problem 1.12 is "No", and the answer to its second part is "Yes" if there are no (inner) models of set theory with large cardinals. To complete the picture, it remains to show that the answer is "No" (to both parts) when large cardinal hypotheses are available. For the following theorem, it suffices for example to assume the consistency of supercompact cardinals, or of so-called *strong cardinals*. More precise large cardinal hypotheses are available in [10].

Theorem 3.3. *(Gitik–Magidor [10]) It is consistent (relative to the consistency of ZFC with an appropriate large cardinal hypothesis) that $2^{\aleph_n} = \aleph_{n+1}$ for all n, and $2^{\aleph_\omega} = \aleph_{\omega+\gamma+1}$, for any prescribed $\gamma < \omega_1$.*

This theorem is related to our questions as follows. As $\aleph_\omega$ is a limit cardinal of cofinality $\aleph_0$, $2^{\aleph_\omega} = (2^{<\aleph_\omega})^{\aleph_0}$. If $2^{\aleph_n} = \aleph_{n+1}$ for all n, then $2^{<\aleph_\omega} = \aleph_\omega$, and thus $2^{\aleph_\omega} = (\aleph_\omega)^{\aleph_0} = 2^{\aleph_0}\cdot\operatorname{cof}([\aleph_\omega]^{\aleph_0}) = \operatorname{cof}([\aleph_\omega]^{\aleph_0})$.[1]

Hechler's forcing $\mathbb{H}$ is a natural forcing notion adding a dominating real, i.e., $d \in \mathbb{N}^{\mathbb{N}}$ such that for each $f \in \mathbb{N}^{\mathbb{N}} \cap V$, where V is the ground model, $f \leq^* d$. $\mathbb{H} = \{(n, f) : n \in \mathbb{N}, f \in \mathbb{N}^{\mathbb{N}}\}$, and $(n, f) \leq (m, g)$ if $n \geq m$, $f \geq g$, and $f(k) = g(k)$ for all $k < m$. If G is $\mathbb{H}$-generic over V, then by a density argument, $d = \bigcup_{(n,f)\in G} f|_{\{1,\dots,n\}} \in \mathbb{N}^{\mathbb{N}}$ is as required. $\mathbb{H}$ is ccc, and thus so is the finite support iteration $P = \langle P_\alpha, \dot{Q}_\alpha : \alpha < \lambda \rangle$, where for each α, P_α forces that $\dot{Q}_\alpha$ is Hechler's forcing.

Theorem 3.4. *It is consistent (relative to the consistency of ZFC with appropriate large cardinal hypotheses) that*

$$\aleph_\omega < \mathfrak{b} = \mathfrak{d} = \aleph_{\omega+1} < \operatorname{cof}(\operatorname{Fin}(\aleph_\omega)^{\mathbb{N}}) = \operatorname{cof}([\aleph_\omega]^{\aleph_0}) = \aleph_{\omega+\gamma+1} = \mathfrak{c},$$

for each prescribed γ *with* $1 \leq \gamma < \aleph_1$.

Proof. Use Theorem 3.3 to produce a model of set theory, V, satisfying $\mathfrak{c} = \aleph_1$ and $\operatorname{cof}([\aleph_\omega]^{\aleph_0}) = \aleph_{\omega+\gamma+1}$.

Let $P := \langle P_\alpha, \dot{Q}_\alpha : \alpha < \aleph_{\omega+1} \rangle$ be the finite support iteration, where for each α, P_α forces that $\dot{Q}_\alpha$ is Hechler's forcing. Let G be P-generic over V, and for each $\alpha < \aleph_{\omega+1}$, let $G_\alpha := G \cap P_\alpha$ be the induced P_α-generic filter over V. For each α, let d_α be the dominating real added by Q_α in stage $\alpha + 1$, so that for each $f \in V[G_\alpha] \cap \mathbb{N}^{\mathbb{N}}$, $f \leq^* d_\alpha$.

As P is ccc, $\operatorname{cof}([\aleph_\omega]^{\aleph_0})$ remains $\aleph_{\omega+\gamma+1}$ in $V[G]$. As $\aleph_{\omega+1}$ has uncountable cofinality, we have that $\mathbb{N}^{\mathbb{N}} \cap V[G] = \bigcup_{\alpha<\aleph_{\omega+1}} \mathbb{N}^{\mathbb{N}} \cap V[G_\alpha]$ [1, Lemma 1.5.7]. It follows that $\{d_\alpha : \alpha < \aleph_{\omega+1}\}$ is dominating in $V[G]$. Moreover, it follows that for each $B \subseteq \mathbb{N}^{\mathbb{N}} \cap V[G]$ with $|B| < \aleph_{\omega+1}$, there is $\alpha < \aleph_{\omega+1}$ such that $B \subseteq \mathbb{N}^{\mathbb{N}} \cap V[G_\alpha]$, and thus B is $\leq^*$-bounded (by d_α). Thus, in $V[G]$, $\mathfrak{b} = \mathfrak{d} = \aleph_{\omega+1}$.

As the Continuum Hypothesis holds in V, $|P| = \aleph_{\omega+1}$, and as P is ccc, the value of $\mathfrak{c}$ in $V[G]$ is at most (by counting nice names [15, Lemma 5.13 in Chapter VII]) $|P|^{\aleph_0} = \aleph_{\omega+1}^{\aleph_0}$, evaluated in V. In V, $\aleph_{\omega+1}^{\aleph_0} \leq (2^{\aleph_\omega})^{\aleph_0} = 2^{\aleph_\omega} = \aleph_{\omega+\gamma+1}$. Thus, in $V[G]$, $\mathfrak{c} \leq \aleph_{\omega+\gamma+1}$. On the other hand, in $V[G]$, as $\aleph_\omega < \mathfrak{d} \leq \mathfrak{c}$, $\aleph_{\omega+\gamma+1} = \operatorname{cof}([\aleph_\omega]^{\aleph_0}) \leq \aleph_\omega^{\aleph_0} \leq \mathfrak{c}^{\aleph_0} = \mathfrak{c}$. □

Remark 3.5. For finite γ, which are sufficient for our purposes, a simplified proof of the Gitik–Magidor Theorem 3.3 is available in Gitik's chapter [9]. Following our proof, Assaf Rinot pointed out to us that starting with a supercompact cardinal (a stronger assumption than that in [9]), one may argue as follows: Start with a model of GCH with κ supercompact. Use Silver forcing to make $2^\kappa = \kappa^{++}$ [12, Theorem 21.4]. Since κ remains measurable, we can use Prikry forcing to make $\operatorname{cof}(\kappa) = \aleph_0$, without adding bounded subsets [12, Theorem 21.10]. Then GCH holds up to κ, and $\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa^{\aleph_0} = 2^\kappa = \kappa^{++}$. Then, continue as in the proof of Theorem 3.4.

4. Comments and open problems

Remarkably, the following problem remains open.

Problem 4.1. (Bonanzinga–Matveev [7]) Is there, consistently, a star-Menger Ψ-space of cardinality $\geq \mathfrak{d}$?

Since Ψ-spaces of cardinality smaller than $\mathfrak{d}$ are strongly star-Menger, the problem asks whether there could be star-Menger Ψ-spaces that are not in fact strongly star-Menger. More importantly, the problem asks whether there may be, consistently, *nontrivial* star-Menger Ψ-spaces, that is, ones whose being star-Menger

[1] For the second equality, count the countable subsets of $\aleph_\omega$ by taking a cofinal family in $[\aleph_\omega]^{\aleph_0}$ and, for each set in this family, take all of its subsets.

does not follow from their cardinality being smaller than $\mathfrak{d}$. By Theorem 1.9, the cardinality of a nontrivial star-Menger Ψ-space cannot be any of the cardinals listed in Theorem 1.10. Thus, $\mathfrak{c} > \aleph_\omega$ in every model witnessing a positive solution of Problem 4.1. It may be worth considering forcing extensions where $\mathfrak{d} = \aleph_1$, $\kappa = \aleph_\omega$, and $\mathfrak{c} = \aleph_{\omega+1}$. Similarly, we have the following problem (to which similar comments apply).

Problem 4.2. Is there, consistently, a star-Hurewicz Ψ-space of cardinality $\geq \mathfrak{b}$?

A topological space X is *star-Rothberger* [13] if for every sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of X, there are elements $U_1 \in \mathcal{U}_1, U_2 \in \mathcal{U}_2, \dots$ such that $X = \bigcup_n \operatorname{star}(U_n, \mathcal{U}_n)$. Arguments similar to ones in Section 2 establish the following theorem.

Theorem 4.3. *Let $\mathcal{A} \subseteq P(\mathbb{N})$ be an almost disjoint family. The following assertions are equivalent:*

(1) *The Isbell–Mrówka space $\Psi(\mathcal{A})$ is star-Rothberger.*
(2) *For each function $A \mapsto f_A$ from $\mathcal{A}$ to $\mathbb{N}^\mathbb{N}$, there are elements $A_1, A_2, \dots \in \mathcal{A}$ such that, for each $A \in \mathcal{A}$, there is n with $(A \setminus f_A(n)) \cap (A_n \setminus f_{A_n}(n)) \neq \emptyset$.* □

The cardinal $\operatorname{cov}(\mathcal{M})$ is the minimal cardinality of a subset of $\mathbb{N}^\mathbb{N}$ that cannot be guessed by a single function (that is, no function is equal infinitely often to each member of the set). It is open whether there is an analogue of Theorems 1.9 and 2.4 for star-Rothberger Ψ-spaces. Ψ-spaces of cardinality smaller than $\operatorname{cov}(\mathcal{M})$ are star-Rothberger, and there is Ψ-space of cardinality $\operatorname{cov}(\mathcal{M})$ that is not star-Rothberger [7].

Problem 4.4. Is there, consistently, an almost disjoint family $\mathcal{A} \subseteq P(\mathbb{N})$ of cardinality $\kappa \geq \operatorname{cov}(\mathcal{M})$ such that $\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa$ and the space $\Psi(\mathcal{A})$ is star-Rothberger?

It is not clear that the cardinals in Theorems 1.2 and 2.4 are not mere artifact of the proofs. Indeed, the proofs exploit the freedom provided by Theorems 2.1 and 2.2. In particular, we have the following problems.

Problem 4.5. What is the minimal cardinal κ such that no Ψ-space of cardinality κ is star-Menger? What is the corresponding cardinal for star-Hurewicz and star-Rothberger Ψ-spaces?

In light of Section 2, it may be possible to prove, using the methods of [17], the following variations of Theorems 1.5 and 1.6

Conjecture 4.6.

(1) *Let D be a closed discrete subspace of a regular strongly star-Hurewicz space. Then the cardinality of D is smaller than the minimal fixed point of the function $\kappa \mapsto \operatorname{cof}([\kappa]^{\aleph_0})$ in the interval $[\mathfrak{b}, \mathfrak{c}]$.*
(2) *Let X be a regular topological space of cardinality κ. If $\operatorname{cof}([\kappa]^{\aleph_0}) = \kappa \geq \mathfrak{b}$, then the space $\mathrm{PR}(X)$ is not star-Hurewicz.*

Motivated by Theorem 1.5, Sakai proposes the following problem.

Problem 4.7. (Sakai) Consider the minimal cardinal number greater than all cardinalities of closed discrete subspaces of regular strongly star-Menger spaces. Is this cardinal equal to the minimal fixed point of the function $\kappa \mapsto \operatorname{cof}([\kappa]^{\aleph_0})$ in the interval $[\mathfrak{d}, \mathfrak{c}]$?

Acknowledgements

I thank Moti Gitik for bringing Theorem 3.3 to my attention and for his useful suggestions, Assaf Rinot for his comment in Remark 3.5, and Masami Sakai and Shir Sivroni for their useful comments. I owe special thanks to Ari Meir Brodsky, whose comments helped improving the presentation of this paper considerably, and to the referee for a detailed and useful report.

A part of the research reported here was conducted during a Sabbatical leave at the Faculty of Mathematics and Computer Science, Weizmann Institute of Science. I thank Gideon Schechtman and the Faculty of Mathematics and Computer Science for their hospitality.

References

[1] T. Bartoszyński, H. Judah, Set Theory: On the Structure of the Real Line, A. K. Peters, Massachusetts, 1995.
[2] A. Blass, Combinatorial cardinal characteristics of the continuum, in: M. Foreman, A. Kanamori, M. Magidor (Eds.), Handbook of Set Theory, Kluwer Academic Publishers, 2010, pp. 395–490.
[3] M. Bonanzinga, F. Cammaroto, L. Kočinac, Star-Hurewicz and related spaces, Appl. Gen. Topol. 5 (2004) 79–89.
[4] E. van Douwen, The integers and topology, in: K. Kunen, J. Vaughan (Eds.), Handbook of Set Theoretic Topology, North-Holland, Amsterdam, 1984, pp. 111–167.
[5] E. van Douwen, G. Reed, A. Roscoe, I. Tree, Star covering properties, Topol. Appl. 39 (1991) 71–103.
[6] C. Chis, M.V. Ferrer, S. Hernández, B. Tsaban, The character of topological groups, via bounded systems, Pontryagin–van Kampen duality and pcf theory, J. Algebra 420 (2014) 86–119.
[7] M. Bonanzinga, M. Matveev, Some covering properties for Ψ-spaces, Mat. Vesn. 61 (2009) 3–11.
[8] M. Gitik, The strength of the failure of the singular cardinal hypothesis, Ann. Pure Appl. Log. 51 (1991) 215–240.
[9] M. Gitik, Prikry-type forcings, in: M. Foreman, A. Kanamori, M. Magidor (Eds.), Handbook of Set Theory, Kluwer Academic Publishers, Dordrecht, 2010, pp. 1351–1448.
[10] M. Gitik, M. Magidor, The singular cardinal hypothesis revisited, in: Set Theory of the Continuum, Berkeley, CA, 1989, in: Math. Sci. Res. Inst. Publ., vol. 26, Springer, New York, 1992, pp. 243–279.
[11] L. Gillman, M. Jerison, Rings of Continuous Functions, Van Nostrand, Princeton, NJ, 1960.
[12] T. Jech, Set Theory, the third millennium edition, Springer-Verlag, 2002.
[13] L. Kočinac, Star-Menger and related spaces, Publ. Math. (Debr.) 55 (1999) 421–431.
[14] L. Kočinac, Star selection principles: a survey, Khayyam J. Math. 1 (2015) 82–106.
[15] K. Kunen, Set Theory: An Introduction to Independence Proofs, Stud. Logic Found. Math., vol. 102, North Holland Publishing Company, 1980.
[16] S. Mrówka, On completely regular spaces, Fundam. Math. 41 (1954) 105–106.
[17] M. Sakai, Star versions of the Menger property, Topol. Appl. 176 (2014) 22–34.
[18] B. Tsaban, Menger's and Hurewicz's problems: solutions from "The Book" and refinements, Contemp. Math. 533 (2011) 211–226.