Annals of Pure and Applied Logic 165 (2014) 1034–1057

Contents lists available at ScienceDirect

Annals of Pure and Applied Logic

www.elsevier.com/locate/apal

# Selective covering properties of product spaces

Arnold W. Miller [a], Boaz Tsaban [b,*,1], Lyubomyr Zdomskyy [c]

[a] *Department of Mathematics, University of Wisconsin–Madison, Van Vleck Hall, 480 Lincoln Drive, Madison, WI 53706-1388, USA*
[b] *Department of Mathematics, Bar-Ilan University, Ramat Gan 5290002, Israel*
[c] *Kurt Gödel Research Center for Mathematical Logic, University of Vienna, Währinger Str. 25, 1090 Vienna, Austria*

ARTICLE INFO

*Article history:*
Received 22 March 2013
Received in revised form 1 January 2014
Accepted 7 January 2014
Available online 22 January 2014

*MSC:*
26A03
03E17
03E75

*Keywords:*
Gerlits–Nagy $\gamma$ property
Menger property
Hurewicz property
Rothberger property
Gerlits–Nagy $(*)$ property
Productively Lindelöf
Product theory
Selection principles
Special sets of real numbers

ABSTRACT

We study the preservation of selective covering properties, including classic ones introduced by Menger, Hurewicz, Rothberger, Gerlits and Nagy, and others, under products with some major families of concentrated sets of reals.
Our methods include the projection method introduced by the authors in an earlier work, as well as several new methods. Some special consequences of our main results are (definitions provided in the paper):

(1) Every product of a concentrated space with a Hurewicz $\mathsf{S}_1(\Gamma, \mathrm{O})$ space satisfies $\mathsf{S}_1(\Gamma, \mathrm{O})$. On the other hand, assuming the Continuum Hypothesis, for each Sierpiński set $S$ there is a Luzin set $L$ such that $L \times S$ can be mapped onto the real line by a Borel function.
(2) Assuming Semifilter Trichotomy, every concentrated space is productively Menger and productively Rothberger.
(3) Every scale set is productively Hurewicz, productively Menger, productively Scheepers, and productively Gerlits–Nagy.
(4) Assuming $\mathfrak{d} = \aleph_1$, every productively Lindelöf space is productively Hurewicz, productively Menger, and productively Scheepers.

A notorious open problem asks whether the additivity of Rothberger's property may be strictly greater than $\mathrm{add}(\mathcal{N})$, the additivity of the ideal of Lebesgue-null sets of reals. We obtain a positive answer, modulo the consistency of Semifilter Trichotomy ($\mathfrak{u} < \mathfrak{g}$) with $\mathrm{cov}(\mathcal{M}) > \aleph_1$.
Our results improve upon and unify a number of results, established earlier by many authors.



* Corresponding author.
*E-mail addresses:* miller@math.wisc.edu (A.W. Miller), tsaban@math.biu.ac.il (B. Tsaban), lzdomsky@logic.univie.ac.at (L. Zdomskyy).
*URLs:* http://www.math.wisc.edu/~miller/ (A.W. Miller), http://www.cs.biu.ac.il/~tsaban (B. Tsaban), http://www.logic.univie.ac.at/~lzdomsky/ (L. Zdomskyy).
[1] Current address: Department of Mathematics, Weizmann Institute of Science, Rehovot 7610001, Israel.

0168-0072/$ – see front matter 
http://dx.doi.org/10.1016/j.apal.2014.01.001

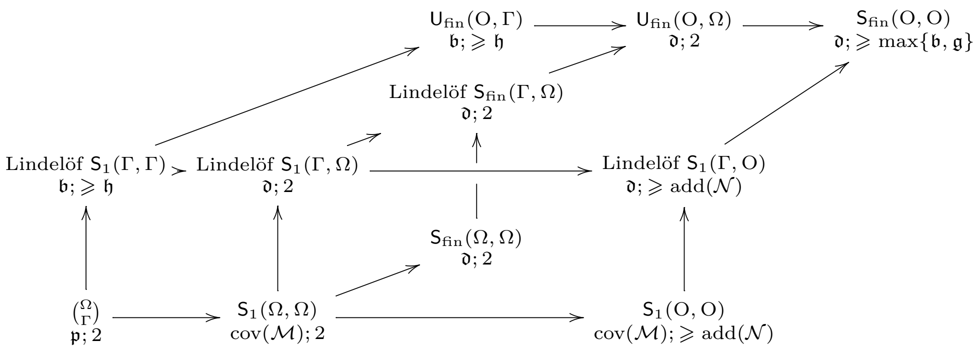


**Fig. 1.** The Scheepers Diagram.

## 1. Introduction

All topological spaces in this paper are assumed, without further mention, to be Tychonoff. Since the results presented here are new even in the case where the spaces are subsets of the real line, readers who wish to do so may assume throughout that we deal with sets of real numbers.

We study selective covering properties of products of topological spaces. Our results, that answer questions concerning classic covering properties, are best perceived in the modern framework of selection principles, to which we provide here a brief introduction.[2] This framework was introduced by Scheepers in [27] to study, in a uniform manner, a variety of properties introduced in different mathematical disciplines, since the early 1920's, by Menger, Hurewicz, Rothberger, Gerlits and Nagy, and others.

Let $X$ be a topological space. We say that $\mathcal{U}$ is a *cover* of $X$ if $X = \bigcup \mathcal{U}$, but $X \notin \mathcal{U}$. Often, $X$ is considered as a subspace of another space $Y$, and in this case we always consider covers of $X$ by subsets of $Y$, and require instead that no member of the cover contains $X$. Let $\mathrm{O}(X)$ be the family of open covers of $X$. Define the following subfamilies of $\mathrm{O}(X)$: $\mathcal{U} \in \Omega(X)$ if each finite subset of $X$ is contained in some member of $\mathcal{U}$. $\mathcal{U} \in \Gamma(X)$ if $\mathcal{U}$ is infinite, and each element of $X$ is contained in all but finitely many members of $\mathcal{U}$.

Some of the following statements may hold for families $\mathscr{A}$ and $\mathscr{B}$ of covers of $X$.

- $\binom{\mathscr{A}}{\mathscr{B}}$ Each member of $\mathscr{A}$ contains a member of $\mathscr{B}$.
- $\mathsf{S}_1(\mathscr{A}, \mathscr{B})$ For each sequence $\langle \mathcal{U}_n \in \mathscr{A}\colon n \in \mathbb{N}\rangle$, there is a selection $\langle U_n \in \mathcal{U}_n\colon n \in \mathbb{N}\rangle$ such that $\{U_n\colon n \in \mathbb{N}\} \in \mathscr{B}$.
- $\mathsf{S}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$ For each sequence $\langle \mathcal{U}_n \in \mathscr{A}\colon n \in \mathbb{N}\rangle$, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n\colon n \in \mathbb{N}\rangle$ such that $\bigcup_n \mathcal{F}_n \in \mathscr{B}$.
- $\mathsf{U}_{\mathrm{fin}}(\mathscr{A}, \mathscr{B})$ For each sequence $\langle \mathcal{U}_n \in \mathscr{A}\colon n \in \mathbb{N}\rangle$, where no $\mathcal{U}_n$ contains a finite subcover, there is a selection of finite sets $\langle \mathcal{F}_n \subseteq \mathcal{U}_n\colon n \in \mathbb{N}\rangle$ such that $\{\bigcup \mathcal{F}_n\colon n \in \mathbb{N}\} \in \mathscr{B}$.

We say, e.g., that $X$ satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ if the statement $\mathsf{S}_1(\mathrm{O}(X), \mathrm{O}(X))$ holds. This way, $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ is a property (or a class) of topological spaces, and similarly for all other statements and families of covers. In the realm of Lindelöf spaces,[3] each nontrivial property among these properties, where $\mathscr{A}, \mathscr{B}$ range over $\mathrm{O}, \Omega, \Gamma$, is equivalent to one in Fig. 1 [27,14]. In this diagram, an arrow denotes implication.

[2] This introduction is adopted from [21]. Extended introductions to this field are available in [16,28,31].
[3] Indeed, all properties in the Scheepers Diagram (Fig. 1), except for those having $\Gamma$ in the first argument, imply being Lindelöf.

The extremal properties in this diagram are classic and were introduced by Menger ($\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$, under a difference guise), Hurewicz ($\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$), Rothberger ($\mathsf{S}_1(\mathrm{O},\mathrm{O})$), and Gerlits and Nagy ($\binom{\Omega}{\Gamma}$). The other ones were introduced and studied more recently, by many authors.

In this diagram, we indicate below each class $P$ its *critical cardinality* $\mathrm{non}(P)$, the minimal cardinality of a space not in the class, and its (provable) *additivity number* $\mathrm{add}(P)$, the minimal number of spaces possessing this property, whose union does *not* have this property. These cardinals are all combinatorial cardinal characteristics of the continuum, details about which are available in [12]. Here, $\mathcal{M},\mathcal{N}$ are the families of meager (i.e., Baire first category) sets in $\mathbb{R}$ and Lebesgue null sets in $\mathbb{R}$, respectively. In cases where only lower bounds on $\mathrm{add}(P)$ are given, $\mathrm{cf}(\mathrm{non}(P))$ is an upper bound.

On occasions, we will also consider the classes of covers $\mathrm{B}$, $\mathrm{B}_\Omega$ and $\mathrm{B}_\Gamma$, defined as $\mathrm{O}$, $\Omega$ and $\Gamma$ were defined, replacing *open cover* by *countable Borel cover*. The properties thus obtained have rich history of their own [29], and for Lindelöf spaces, the Borel variants of the studied properties are (usually, strictly) stronger than the open ones [29].

Many additional—classic and new—properties were or can be studied in relation to the Scheepers Diagram. Some examples of this kind are provided in the present paper.

The following definition and observation are useful.

**Definition 1.1.** Let $P$ be a property (or class) of topological spaces.[4] A topological space $X$ is *productively* $P$ if $X \times Y$ has the property $P$ for each $Y$ satisfying $P$. $P^{\uparrow}$ is the property of having all finite powers satisfying $P$.

In this notation, $\mathsf{S}_1(\mathrm{O},\mathrm{O})^{\uparrow} = \mathsf{S}_1(\Omega,\Omega)$ [25] and $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})^{\uparrow} = \mathsf{S}_{\mathrm{fin}}(\Omega,\Omega)$ [14]. If $X$ is productively $P$ and the singleton space satisfies $P$, then $X$ satisfies $P$. Moreover, we have the following.

**Lemma 1.2.** *Let $X$ be a productively $P$ topological space. Then*:

(1) *Every finite power of $X$ is productively $P$.*
(2) *$X$ is productively $P^{\uparrow}$.*
(3) *Every finite power of $X$ is productively $P^{\uparrow}$.*

**Proof.** (1) By induction on the power of $X$, $X^k \times Y$ has the property $P$ if $Y$ has it.

(2) Let $Y$ be in $P^{\uparrow}$. For each $k$, $(X \times Y)^k \cong X^k \times Y^k$. Apply (1).

(3) By (1) and (2). □

In particular, if $X$ is productively $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, then it is also productively $\mathsf{S}_1(\Omega,\Omega)$, and similarly for $\mathsf{S}_{\mathrm{fin}}$. Several additional properties in the literature are characterized by having the form $P^{\uparrow}$ for a property $P$ in the Scheepers Diagram, and the same comment applies.

## 2. Concentrated spaces and $\mathsf{S}_1(\Gamma,\Gamma)$

Let $\kappa$ be an uncountable cardinal. Following Besicovitch [9,10], we say that a topological space $X$ is *$\kappa$-concentrated* if there is a countable set $D \subseteq X$ such that $|X \setminus U| < \kappa$ for every open set $U \supseteq D$. Several major examples of families of concentrated spaces will be considered later.

A special case of Theorem 11(3) in Babinkostova–Scheepers [3] is that for each concentrated metric space $C$, if $Y$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $\mathsf{S}_1(\mathrm{O},\mathrm{O})$, then $C \times Y$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$. Theorem 3.1 in the more recent paper [37] implies, in particular, that it suffices to assume that $C$ is a $\mathrm{cov}(\mathcal{M})$-concentrated space.

[4] The property $P$ may or may not stand for "P-space".

Our first observation is that the methods of the paper [37] imply a similar result for $\mathsf{S}_1(\Gamma,\Gamma)$. The proof given here is slightly more general than the one that may be extracted from [37]. Initially, we only stated in the following lemma that $X$ satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$. That in fact $X$ satisfies $\mathsf{S}_1(\Gamma,\Omega)$ was pointed out to us by the referee.

**Lemma 2.1.** *Let a topological space $X$ be a union of less than $\mathrm{cov}(\mathcal{M})$ many $\mathsf{S}_1(\Gamma,\Gamma)$ spaces. Then $X$ satisfies $\mathsf{S}_1(\Gamma,\Omega)$.*

**Proof.** Let $\{X_\alpha\colon \kappa<\mathrm{cov}(\mathcal{M})\}$ be a family of $\mathsf{S}_1(\Gamma,\Gamma)$ spaces. As $\mathsf{S}_1(\Gamma,\Gamma)$ is additive, we may assume that this family is closed under finite unions.

Let $X=\bigcup_{\alpha<\kappa}X_\alpha$, and let $\mathcal{U}_n\in\Gamma(X)$ for all $n$. As $\Gamma(X)$ is closed taking infinite subsets, we may assume that we can enumerate $\mathcal{U}_n=\{U^n_m\colon m\in\mathbb{N}\}$ for all $n$.

For each $\alpha<\kappa$, let $f_\alpha\in\mathbb{N}^\mathbb{N}$ be such that $\{U^n_{f_\alpha(n)}\colon n\in\mathbb{N}\}\in\Gamma(X_\alpha)$. As $\kappa<\mathrm{cov}(\mathcal{M})$, there is $f\in\mathbb{N}^\mathbb{N}$ such that, for each $\alpha<\kappa$, $f(n)=f_\alpha(n)$ for infinitely many $n$. Then $\{U^n_{f(n)}\colon n\in\mathbb{N}\}\in\Omega(X)$. □

The method used in the following proof was introduced in [37]. Since this method is used frequently in the present paper, we name it *the projection method*.

**Theorem 2.2.** *Let $C$ be a $\mathrm{cov}(\mathcal{M})$-concentrated space. For each Lindelöf $\mathsf{S}_1(\Gamma,\Gamma)$ space $Y$, $X\times Y$ satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$.*

**Proof.** Let $C$ be $\mathrm{cov}(\mathcal{M})$-concentrated on some countable set $D\subseteq C$. Let $Y$ be a Lindelöf $\mathsf{S}_1(\Gamma,\Gamma)$ space. Let $K$ be a compact space containing $C$ as a subspace. For each $n$, let $\mathcal{U}_n\in\Gamma(C\times Y)$, where the elements of $\mathcal{U}_n$ are open in $K\times Y$.

As $D$ is countable, $D\times Y$ satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$ (Lemma 2.1). Pick $U_n\in\mathcal{U}_n$, $n\in\mathbb{N}$, such that $D\times Y\subseteq U:=\bigcup_n U_n$.

The Hurewicz property $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ is preserved by products with compact spaces, moving to closed subspaces, and continuous images [14]. Since $Y$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $K$ is compact, $K\times Y$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. Thus, so does $K\times Y\setminus U$. It follows that the projection $H$ of $(K\times Y)\setminus U$ on the first coordinate, satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. Note that

$$(K\setminus H)\times Y\subseteq U.$$

The argument in the proof of [14, Theorem 5.7] generalizes to arbitrary (Tychonoff) spaces, to show that for $H,F$ disjoint subspaces of a space $K$ with $H$ $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$, and $F$ $\mathrm{F}_\sigma$, there is a $\mathrm{G}_\delta$ set $G\subseteq K$ such that $G\supseteq F$ and $H\cap G=\emptyset$.

Let $G$ be a $\mathrm{G}_\delta$ subset of $K$ such that $D\subseteq G$ and $H\cap G=\emptyset$. As $C$ is $\mathrm{cov}(\mathcal{M})$-concentrated on $D$, $C\setminus G$ is a countable increasing union of sets of cardinality $<\mathrm{cov}(\mathcal{M})$. By Lemma 2.1 and the fact that $\mathsf{S}_1(\Gamma,\mathrm{O})$ is preserved under countable unions (or, alternatively, as $\mathrm{cov}(\mathcal{M})$ has uncountable cofinality), $(C\setminus G)\times Y$ satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$. Take $V_n\in\mathcal{U}_n$, $n\in\mathbb{N}$, such that $(C\setminus G)\times Y\subseteq\bigcup_n V_n$. Then

$$C\times Y\subseteq\bigcup_{n\in\mathbb{N}}(U_n\cup V_n).$$

We have picked two sets (instead of one) from each cover $\mathcal{U}_n$, but this is fine (e.g., [34, Appendix A]). □

The methods of [37] also imply the following, more general result. Since the proof is similar to that of Theorem 3.3 in [37] and we are not going to use this result here, we omit the proof.

**Definition 2.3.** Let $\kappa$ be an infinite cardinal number. Let $\mathscr{C}_0(\kappa)$ be the family of spaces of cardinality $<\kappa$. For successor ordinals $\alpha+1$, let $C \in \mathscr{C}_{\alpha+1}(\kappa)$ if:

(1) either there is a countable $D \subseteq C$ with $C \setminus U \in \mathscr{C}_\alpha(\kappa)$ for all open $U \supseteq D$;
(2) or $C$ is a union of less than $\mathrm{cf}(\kappa)$ members of $\mathscr{C}_\alpha(\kappa)$.

For limit ordinals $\alpha$, let $\mathscr{C}_\alpha(\kappa) = \bigcup_{\beta<\alpha} \mathscr{C}_\beta(\kappa)$.

By the Scheepers Diagram, $\mathrm{add}(\mathcal{N}) \leqslant \mathrm{add}(\mathsf{S}_1(\Gamma, \mathrm{O}))$.

**Theorem 2.4.** *The product of each member of $\mathscr{C}_{\mathrm{add}(\mathcal{N})}(\mathrm{cov}(\mathcal{M}))$ with every Lindelöf $\mathsf{S}_1(\Gamma,\Gamma)$ space satisfies $\mathsf{S}_1(\Gamma, \mathrm{O})$.* □

**Definition 2.5.** Let $P, Q$ be classes of spaces, each containing all one-element spaces and closed under homeomorphic images. $(P,Q)^\times$ is the class of all spaces $X$ such that, for each $Y$ in $P$, $X \times Y$ is in $Q$.

By Lemma 2.1, $\mathrm{cov}(\mathcal{M}) \leqslant \mathrm{non}((\mathsf{S}_1(\Gamma,\Gamma), \mathsf{S}_1(\Gamma,\mathrm{O}))^\times)$. Theorem 2.4 holds, more generally, for $\mathscr{C}_{\mathrm{add}(\mathsf{S}_1(\Gamma,\mathrm{O}))}(\mathrm{non}((\text{Lindelöf } \mathsf{S}_1(\Gamma,\Gamma), \mathsf{S}_1(\Gamma,\mathrm{O}))^\times))$.

**Problem 2.6.** Is $\mathrm{non}((\text{Lindelöf } \mathsf{S}_1(\Gamma,\Gamma), \mathsf{S}_1(\Gamma,\mathrm{O}))^\times) = \mathfrak{d}$?

## 3. Concentrated sets and the conjunction of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $\mathsf{S}_1(\Gamma,\mathrm{O})$

In this section, we consider the conjunction of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $\mathsf{S}_1(\Gamma,\mathrm{O})$. This class is larger than Lindelöf $\mathsf{S}_1(\Gamma,\Gamma)$. The definition of $\mathfrak{b}$-*scale set* is given in Section 6. For the present purpose, it suffices to know their following properties (cf. [34]): $\mathfrak{b}$-scale sets are subspaces of $\mathbb{R}$, of cardinality $\mathfrak{b}$, that can be constructed outright in ZFC. They are $\mathfrak{b}$-concentrated, and as such satisfy $\mathsf{S}_1(\Gamma,\mathrm{O})$, and they satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. The following results are known.

**Theorem 3.1.**

(1) *Every $\mathfrak{b}$-scale set satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $\mathsf{S}_1(\Gamma,\mathrm{O})$ [7] (cf. [34]).*
(2) *Consistently, no set of reals of cardinality $\mathfrak{b}$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ [20].*
(3) *The Continuum Hypothesis implies that there is a $\mathfrak{b}$-scale set not satisfying $\mathsf{S}_1(\Gamma,\Gamma)$ [24].*

We will show that the conjunction of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $\mathsf{S}_1(\Gamma,\mathrm{O})$ can be expressed as a standard selective property. A countable cover $\mathcal{U}$ of a space $X$ is in $\daleth(\Gamma)$ [26] if for each (equivalently, some) bijective enumeration $\mathcal{U} = \{U_n \colon n \in \mathbb{N}\}$, there is an increasing $h \in \mathbb{N}^\mathbb{N}$ such that, for each $x \in X$,

$$x \in \bigcup_{k=h(n)}^{h(n+1)-1} U_k$$

for all but finitely many $n$. In [17] it is shown that $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma) \cap \mathsf{S}_1(\mathrm{O},\mathrm{O}) = \mathsf{S}_1(\Omega, \daleth(\Gamma))$.

**Proposition 3.2.** $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma) \cap \mathsf{S}_1(\Gamma,\mathrm{O}) =$ *Lindelöf* $\mathsf{S}_1(\Gamma, \daleth(\Gamma))$.

**Proof.** ($\Rightarrow$) $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ implies that every countable open cover is in $\daleth(\Gamma)$ [17].

($\Leftarrow$) It suffices to prove that $\mathsf{S}_1(\Gamma, \daleth(\Gamma))$ implies $\mathsf{U}_{\mathrm{fin}}(\Gamma, \Gamma)$. Assume that $X$ satisfies $\mathsf{S}_1(\Gamma, \daleth(\Gamma))$, and let $\mathcal{U}_n \in \Gamma$ for all $n$. We may assume that the covers $\mathcal{U}_n$ get finer with $n$. Apply $\mathsf{S}_1(\Gamma, \daleth(\Gamma))$ to obtain $U_n \in \mathcal{U}_n$, $n \in \mathbb{N}$, and an increasing $h \in \mathbb{N}^{\mathbb{N}}$ such that, for each $x \in X$,

$$x \in \bigcup_{k=h(n)}^{h(n+1)-1} U_k$$

for all but finitely many $n$. Since the covers $\mathcal{U}_n$ get finer with $n$, there is for each $n$ a finite set $\mathcal{F}_{h(n)} \subseteq \mathcal{U}_{h(n)}$ such that

$$\bigcup_{k=h(n)}^{h(n+1)-1} U_k \subseteq \bigcup \mathcal{F}_{h(n)}.$$

For $n$ not in the image of $h$ chose $\mathcal{F}_n = \emptyset$. $\square$

**Lemma 3.3.** *Let a space $X$ be a union of less than $\mathrm{cov}(\mathcal{M})$ many $\mathsf{S}_1(\Gamma, \daleth(\Gamma))$ spaces. Then $X$ satisfies $\mathsf{S}_1(\Gamma, \mathrm{O})$.*

**Proof.** The proof is similar to that of Theorem 2.2 in [37]. We provide it, with the necessary changes, for completeness.

Let $\kappa < \mathrm{cov}(\mathcal{M})$. Assume that, for each $\alpha < \kappa$, $X_\alpha$ satisfies $\mathsf{S}_1(\Gamma, \daleth(\Gamma))$, and $X = \bigcup_{\alpha<\kappa} X_\alpha$. Let $\mathcal{U}_n \in \Gamma(X)$ for all $n$. We may assume that each $\mathcal{U}_n$ is countable, and enumerate $\mathcal{U}_n = \{U^n_m\colon m \in \mathbb{N}\}$. For each $\alpha$, as $X_\alpha$ satisfies $\mathsf{S}_1(\Gamma, \daleth(\Gamma))$, there are $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ and an increasing $h_\alpha \in \mathbb{N}^{\mathbb{N}}$ such that, for each $x \in X_\alpha$,

$$x \in \bigcup_{k=h_\alpha(n)}^{h_\alpha(n+1)-1} U^k_{f_\alpha(k)}$$

for all but finitely many $n$.

Since $\kappa < \mathrm{cov}(\mathcal{M}) \leqslant \mathfrak{d}$ [12], there is an increasing $h \in \mathbb{N}^{\mathbb{N}}$ such that, for each $\alpha < \kappa$, the set

$$I_\alpha = \bigl\{n\colon \bigl[h_\alpha(n), h_\alpha(n+1)\bigr) \subseteq \bigl[h(n), h(n+1)\bigr)\bigr\}$$

is infinite [12]. For each $\alpha < \kappa$, define

$$g_\alpha \in \prod_{n \in I_\alpha} \mathbb{N}^{[h(n),h(n+1))}$$

by $g_\alpha(n) = f_\alpha \restriction [h(n), h(n+1))$ for all $n \in I_\alpha$. As $\kappa < \mathrm{cov}(\mathcal{M})$, by Lemma 2.4.2(3) in [5], there is $g \in \prod_n \mathbb{N}^{[h(n),h(n+1))}$ guessing all functions $g_\alpha$, that is, for each $\alpha < \kappa$, $g(n) = g_\alpha(n)$ for infinitely many $n \in I_\alpha$ [12]. Define $f \in \mathbb{N}^{\mathbb{N}}$ by $f(k) = g(n)(k)$, where $n$ is the one with $k \in [h(n), h(n+1))$. Then $\{U^n_{f(n)}\colon n \in \mathbb{N}\} \in \mathrm{O}(X)$.

Indeed, let $x \in X$. Pick $\alpha < \kappa$ with $x \in X_\alpha$. Pick $m$ such that, for all $n > m$, $x \in \bigcup_{k=h_\alpha(n)}^{h_\alpha(n+1)-1} U^k_{f_\alpha(k)}$. Pick $n \in I_\alpha$ such that $n > m$ and $g(n) = g_\alpha(n)$. Then

$$x \in \bigcup_{k=h_\alpha(n)}^{h_\alpha(n+1)-1} U^k_{f_\alpha(k)} \subseteq \bigcup_{k=h(n)}^{h(n+1)-1} U^k_{f_\alpha(k)} = \bigcup_{k=h(n)}^{h(n+1)-1} U^k_{f(k)}. \qquad \square$$

**Corollary 3.4.** $\mathrm{add}(\mathcal{M}) \leqslant \mathrm{add}(\mathsf{S}_1(\Gamma, \daleth(\Gamma))) \leqslant \mathfrak{b}$.

**Proof.** The second inequality follows from $\mathrm{non}(\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma))=\mathfrak{b}$.

First inequality: $\mathrm{add}(\mathcal{M})=\min\{\mathfrak{b},\mathrm{cov}(\mathcal{M})\}$, and $\mathrm{add}(\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma))=\mathfrak{b}$. Apply Lemma 3.3. □

We obtain the following generalization of Theorem 2.2.

**Theorem 3.5.** *Let $C$ be a $\mathrm{cov}(\mathcal{M})$-concentrated space. For each space $Y$ satisfying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $\mathsf{S}_1(\Gamma,\mathrm{O})$, $X\times Y$ satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$.* □

**Proof.** Similar to the proof of Theorem 2.2, using Lemma 3.3. □

Similarly, we have the following.

**Theorem 3.6.** *The product of each member of $\mathscr{C}_{\mathrm{add}(\mathcal{N})}(\mathrm{cov}(\mathcal{M}))$ with every Lindelöf $\mathsf{S}_1(\Gamma,\daleth(\Gamma))$ space satisfies $\mathsf{S}_1(\Gamma,\mathrm{O})$.* □

By Lemma 3.3, $\mathrm{cov}(\mathcal{M})\leqslant\mathrm{non}((\mathsf{S}_1(\Gamma,\daleth(\Gamma)),\mathsf{S}_1(\Gamma,\mathrm{O}))^{\times})$. Theorem 3.6 holds, more generally, for $\mathscr{C}_{\mathrm{add}(\mathsf{S}_1(\Gamma,\mathrm{O}))}(\mathrm{non}((\text{Lindelöf }\mathsf{S}_1(\Gamma,\daleth(\Gamma)),\mathsf{S}_1(\Gamma,\mathrm{O}))^{\times}))$. Under mild hypotheses on a family $\mathscr{A}$ of covers, the results proved here apply to $\mathsf{S}_1(\mathscr{A},\mathrm{O})$ for all $\mathscr{A}$. The hypotheses on $\mathscr{A}$, which can be extracted from the proofs, are satisfied by all major types of covers in the context of selection principles.

**Problem 3.7.** Is $\mathrm{add}(\mathsf{S}_1(\Gamma,\daleth(\Gamma)))=\mathfrak{b}$?

## 4. Concentrated sets and coherence of filters

For $a\in[\mathbb{N}]^{\infty}$ and an increasing $h\in\mathbb{N}^{\mathbb{N}}$, define

$$a/h=\bigl\{n\colon\ a\cap\bigl[h(n),h(n+1)\bigr)\neq\emptyset\bigr\}.$$

For $S\subseteq[\mathbb{N}]^{\infty}$, define $S/h=\{a/h\colon\ a\in S\}$.

### *4.1. Assuming NCF*

*NCF* (*near coherence of filters*) is the assertion that, for each pair of nonprincipal ultrafilters $\mathcal{U}$ and $\mathcal{V}$, there is an increasing $h\in\mathbb{N}^{\mathbb{N}}$ such that $\mathcal{U}/h=\mathcal{V}/h$. The basic facts about NCF used here are available, e.g., in [11,13].

Henceforth, we use the convenient notation

$$U^n_{\leqslant g(n)}:=\bigcup_{m=1}^{g(n)}U^n_m.$$

We are indebted to Taras Banakh for proposing the following observation and its proof idea.

**Theorem 4.1** (NCF). *For a space $X$, the following assertions are equivalent*:

1. *$X$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)$.*
2. *Whenever $X\subseteq G\subseteq K$, with $K$ compact and $G$ $G_\delta$ in $K$, there are $\kappa<\mathfrak{d}$ and compact sets $K_\alpha\subseteq K$, $\alpha<\kappa$, such that $X\subseteq\bigcup_{\alpha<\kappa}K_\alpha\subseteq G$.*

*Moreover, the implication (2) ⇒ (1) holds in ZFC, and in the implication (1) ⇒ (2), we may take $\kappa=\mathfrak{u}$.*

**Proof.** (1) ⇒ (2): Since $X$ is Lindelöf, for each open set $U$ containing $X$, there are open sets $U_m$, $m \in \mathbb{N}$, such that $X \subseteq \bigcup_m U_m \subseteq \bigcup_m \overline{U_m} \subseteq U$. Let $G = \bigcap_n U_n$ with each $U_n$ open in $K$. For each $n$, let $U^n_m$, $m \in \mathbb{N}$, be such that

$$X \subseteq \bigcup_{m\in\mathbb{N}} U^n_m \subseteq \bigcup_{m\in\mathbb{N}} \overline{U^n_m} \subseteq U_n,$$

and such that the covers $\{U^n_m\colon\ m \in \mathbb{N}\}$ of $X$ get finer with $n$. We may assume that none of these covers contains a finite subcover of $X$.

Apply $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)$ to obtain an increasing $f \in \mathbb{N}^{\mathbb{N}}$ such that $\{U^n_{\leqslant f(n)}\colon\ n \in \mathbb{N}\} \in \Omega(X)$. For each $x \in X$, let

$$f_x(n) = \min\{m \geqslant f_x(n-1)\colon\ x \in U^n_m\}$$

for all $n$. The family of all sets $\{n\colon\ f_x(n) \leqslant f(n)\}$, $x \in X$, is centered. Extend it to a nonprincipal ultrafilter $\mathcal{U}$ on $\mathbb{N}$. Then $\{f_x\colon\ x \in X\}$ is $\leqslant_{\mathcal{U}}$-bounded.

Let $\mathcal{V}$ be an ultrafilter with base of size $\mathfrak{u}$. By NCF, there is an increasing $h \in \mathbb{N}^{\mathbb{N}}$ such that $\mathcal{U}/h = \mathcal{V}/h$. We claim that $\{f_x\colon\ x \in X\}$ is $\leqslant_{\mathcal{V}}$-bounded. Indeed, $f$ is an increasing $\leqslant_{\mathcal{U}}$-bound for $\{f_x\colon\ x \in X\}$. Define $g(n) = f(h(n+1))$ for all $n$. For each $n \in \{k\colon\ f_x(k) \leqslant f(k)\}/h$, fix $k \in \{k\colon\ f_x(k) \leqslant f(k)\} \cap [h(n), h(n+1))$, then

$$f_x(n) \leqslant f_x\bigl(h(n)\bigr) \leqslant f_x(k) \leqslant f(k) \leqslant f\bigl(h(n+1)\bigr) = g(n).$$

Let $\{A_\alpha\colon\ \alpha < \mathfrak{u}\}$ be a base for $\mathcal{V}$. For each $\alpha < \mathfrak{u}$, let

$$K_\alpha = \bigcap_{n\in A_\alpha} \overline{U^n_{\leqslant g(n)}}.$$

Then each $K_\alpha$ is compact, and $X \subseteq \bigcup_{\alpha<\mathfrak{u}} K_\alpha$. NCF implies that $\mathfrak{u} < \mathfrak{d}$.

(2) ⇒ (1): For each $n$, let $\{U^n_m\colon\ m \in \mathbb{N}\}$ be an open cover of $X$. Let $K$ be a compact space containing $X$. We may assume that each set $U^n_m$ is open in $K$. Let

$$G = \bigcap_{n\in\mathbb{N}} \bigcup_{m\in\mathbb{N}} U^n_m.$$

Then $G$ is $\mathrm{G}_\delta$ in $K$, and $X \subseteq G$. Let $\kappa < \mathfrak{d}$ and $K_\alpha \subseteq K$, $\alpha < \kappa$, be compact sets with $X \subseteq \bigcup_{\alpha<\kappa} K_\alpha \subseteq G$. For each $\alpha < \kappa$, let $f_\alpha \in \mathbb{N}^{\mathbb{N}}$ be such that

$$K_\alpha \subseteq \bigcap_n U^n_{\leqslant f_\alpha(n)}.$$

Let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness that $\{f_\alpha\colon\ \alpha < \kappa\}$ is not finitely dominating. Then $\{U^n_{\leqslant g(n)}\colon\ n \in \mathbb{N}\}$ is in $\Omega(X)$. □

In [8] it was proved that, if NCF holds, then $\mathfrak{b}, \mathfrak{g} \leqslant \mathrm{add}(\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega))$. We obtain an optimal version of this result.

**Corollary 4.2** (NCF). $\mathrm{add}(\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)) = \mathfrak{d}$.

**Proof.** By Theorem 4.1. □

**Theorem 4.3** (NCF). *Let $C$ be a $\mathfrak{d}$-concentrated space. For each $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)$ space $Y$, $X \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.*

**Proof.** We use the projection method.

Assume that there is a countable $D \subseteq C$ with $|C \setminus U| < \mathfrak{d}$ for all open $U \supseteq D$. Let $Y$ be a $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$ space. Let $K$ be a compact space containing $C$ as a subspace. Let $\mathcal{U}_n$, $n \in \mathbb{N}$, be covers of $C \times Y$ by sets open in $K \times Y$.

As $D \times Y$ satisfies $\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $D \times Y \subseteq U := \bigcup_n \bigcup \mathcal{F}_n$.

The projection $S$ of $(K \times Y) \setminus U$ on the first coordinate, satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$. By Theorem 4.1, there are compact sets $K_\alpha$, $\alpha < \mathfrak{u}$, such that

$$S \subseteq \bigcup_{\alpha<\mathfrak{u}} K_\alpha \subseteq K \setminus D.$$

As $C$ is $\mathfrak{d}$-concentrated on $D$, $|C \cap K_\alpha| < \mathfrak{d}$ for all $\alpha < \mathfrak{u}$. By NCF, $\mathfrak{u} < \mathfrak{d}$ and $\mathfrak{d}$ regular. It follows that

$$\left| C \cap \bigcup_{\alpha<\mathfrak{u}} K_\alpha \right| < \mathfrak{d}.$$

By Corollary 4.2, $(C \cap S) \times Y \subseteq (C \cap \bigcup_{\alpha<\mathfrak{u}} K_\alpha) \times Y$ and the latter space satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$. In particular, there are finite $\mathcal{F}'_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $(C \cap S) \times Y \subseteq \bigcup_n \bigcup \mathcal{F}'_n$. Then

$$C \times Y \subseteq \bigcup_{n\in\mathbb{N}} \bigcup (\mathcal{F}_n \cup \mathcal{F}'_n). \qquad \square$$

### *4.2. Assuming* $\mathfrak{u} < \mathfrak{g}$

The axiom $\mathfrak{u} < \mathfrak{g}$ is stronger than NCF [18].

**Theorem 4.4** ($\mathfrak{u} < \mathfrak{g}$). *For a space $X$, the following assertions are equivalent*:

(1) *$X$ satisfies* $\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$.
(2) *Whenever $X \subseteq G \subseteq K$, with $K$ compact and $G$ $G_\delta$ in $K$, there are compact sets $K_\alpha \subseteq K$, $\alpha < \mathfrak{u}$, such that $X \subseteq \bigcup_{\alpha<\mathfrak{u}} K_\alpha \subseteq G$.*

**Proof.** (2) ⇒ (1): As $\mathfrak{u} < \mathfrak{g} \leqslant \mathfrak{d}$, Theorem 4.1 implies that $X$ satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$.

(1) ⇒ (2): $\mathfrak{u} < \mathfrak{g}$ implies that $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega) = \mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$ [38] (cf. [36]). Apply Theorem 4.1. $\square$

**Corollary 4.5** ($\mathfrak{u} < \mathfrak{g}$). $\mathrm{add}(\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})) = \mathfrak{d}$.

**Proof.** By Corollary 4.2, using that $\mathfrak{u} < \mathfrak{g}$ implies that $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega) = \mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$. $\square$

**Definition 4.6.** Let $\mathscr{K}_\mathfrak{d}$ be the smallest (with respect to inclusion) class of topological spaces with the following properties:

(1) Every singleton space is in $\mathscr{K}_\mathfrak{d}$.
(2) $\mathscr{K}_\mathfrak{d}$ is closed under unions of less than $\mathfrak{d}$ elements.
(3) If there is a countable $D \subseteq C$ with $C \setminus U \in \mathscr{K}_\mathfrak{d}$ for all open $U \supseteq D$, then $C \in \mathscr{K}_\mathfrak{d}$.

Notice that every $\mathfrak{d}$-concentrated space is in $\mathscr{K}_\mathfrak{d}$.

**Theorem 4.7** ($\mathfrak{u} < \mathfrak{g}$). *Every member of $\mathscr{K}_\mathfrak{d}$ is productively* $\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$ *and productively* $\mathsf{S}_{\text{fin}}(\Omega, \Omega)$.

**Proof.** It suffices to prove the first assertion. We use the projection method, and argue by induction on the structure of $\mathscr{K}_{\mathfrak{d}}$, as defined in Definition 4.6. Case (1) in this definition is trivial, and Case (2) follows from Corollary 4.5. We treat Case (3).

Assume that there is a countable $D \subseteq C$ with $C \setminus U$ productively $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ for all open $U \supseteq D$. Let $Y$ be an $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ space. Let $K$ be a compact space containing $C$ as a subspace. Let $\mathcal{U}_n$, $n \in \mathbb{N}$, be covers of $C \times Y$ by sets open in $K \times Y$.

As $D \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, there are finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $D \times Y \subseteq U := \bigcup_n \bigcup \mathcal{F}_n$.

The projection $M$ of $(K \times Y) \setminus U$ on the first coordinate, satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. By Theorem 4.4, there are compact sets $K_\alpha$, $\alpha < \mathfrak{u}$, such that

$$M \subseteq \bigcup_{\alpha < \mathfrak{u}} K_\alpha \subseteq K \setminus D.$$

Let $\alpha < \mathfrak{u}$. As $K_\alpha \cap D = \emptyset$, we have by the induction hypothesis that $(C \cap K_\alpha) \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. By Corollary 4.5,

$$\bigcup_{\alpha < \mathfrak{u}} (C \cap K_\alpha) \times Y \subseteq (C \cap M) \times Y$$

satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Take finite $\mathcal{F}'_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $(C \setminus G) \times Y \subseteq \bigcup_n \bigcup \mathcal{F}'_n$. Then

$$C \times Y \subseteq \bigcup_{n \in \mathbb{N}} \bigcup (\mathcal{F}_n \cup \mathcal{F}'_n). \qquad \square$$

A notorious open problem asks whether, consistently, $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ is closed under finite products. By Theorem 4.7, a positive answer to the following problem would settle this problem in the affirmative. The *superperfect set model* is the model obtained by an $\aleph_2$ stage countable support iteration of superperfect trees forcing over a model of GCH. In this model, $\mathfrak{u} < \mathfrak{g}$. The values of the combinatorial cardinal characteristics of the continuum in this model [12] imply that there are no generalized (in any relevant sense) Luzin or Sierpiński sets there (see Section 5 for the definitions). Consequently, in the superperfect set model, the only known spaces satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ are those in $\mathscr{K}_{\mathfrak{d}}$.

**Problem 4.8.** Is $\mathscr{K}_{\mathfrak{d}} = \mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ in the superperfect set model?

We conclude this section with analogous results for Rothberger's property $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. The hypothesis $\mathfrak{u} < \mathfrak{g}$ implies that every $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ space is $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ [38] (cf. [36]), and therefore that $\mathsf{S}_1(\Omega, \daleth(\Gamma)) = \mathsf{S}_1(\mathrm{O}, \mathrm{O})$.

**Definition 4.9.** Let $\mathscr{C}_{\mathrm{cov}(\mathcal{M})}$ be the smallest (with respect to inclusion) class of topological spaces with the following properties:

(1) Every singleton space is in $\mathscr{C}_{\mathrm{cov}(\mathcal{M})}$.
(2) $\mathscr{C}_{\mathrm{cov}(\mathcal{M})}$ is closed under unions of less than $\mathrm{cov}(\mathcal{M})$ elements.
(3) If there is a countable $D \subseteq C$ with $C \setminus U \in \mathscr{C}_{\mathrm{cov}(\mathcal{M})}$ for all open $U \supseteq D$, then $C \in \mathscr{C}_{\mathrm{cov}(\mathcal{M})}$.

Notice that every $\mathrm{cov}(\mathcal{M})$-concentrated space is in $\mathscr{C}_{\mathrm{cov}(\mathcal{M})}$. Using the above methods, we obtain the following.

**Theorem 4.10** ($\mathfrak{u} < \mathfrak{g}$). *Every member of* $\mathscr{C}_{\mathrm{cov}(\mathcal{M})}$ *is productively* $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$ *and productively* $\mathsf{S}_1(\Omega, \Omega)$.

**Proof.** Assuming $\mathfrak{u} < \mathfrak{g}$, since $\mathsf{S}_1(\Omega, \daleth(\Gamma)) = \mathsf{S}_1(\mathrm{O}, \mathrm{O})$, we have by Theorem 2.3 of [37] that

$$\begin{aligned}\operatorname{add}\bigl(\mathsf{S}_1(\mathrm{O},\mathrm{O})\bigr) &= \operatorname{add}\bigl(\mathsf{S}_1\bigl(\Omega,\daleth(\Gamma)\bigr)\bigr) = \operatorname{add}(\mathcal{M}) = \operatorname{non}\bigl(\mathsf{S}_1\bigl(\Omega,\daleth(\Gamma)\bigr)\bigr)\\ &= \operatorname{non}\bigl(\mathsf{S}_1(\mathrm{O},\mathrm{O})\bigr) = \operatorname{cov}(\mathcal{M}).\end{aligned}$$

The rest follows from the projection method, as in the proof of Theorem 4.7. □

A notorious open problem, due to Bartoszyński and Judah [4], asks whether, consistently, $\operatorname{add}(\mathsf{S}_1(\mathrm{O},\mathrm{O}))$ or $\operatorname{add}(\mathsf{S}_1(\mathrm{B},\mathrm{B}))$ may be greater than $\operatorname{add}(\mathcal{N})$. We do not know whether the hypothesis in the following theorem is consistent (it is provable that $\operatorname{cov}(\mathcal{M}) \leqslant \mathfrak{u}$, though), but once such a consistency result is established, we would obtain a solution of this problem.

**Theorem 4.11** ($\aleph_1 < \operatorname{cov}(\mathcal{M}) \leqslant \mathfrak{u} < \mathfrak{g}$).

$$\aleph_1 = \operatorname{add}(\mathcal{N}) < \operatorname{cf}\bigl(\operatorname{cov}(\mathcal{M})\bigr) = \operatorname{cov}(\mathcal{M}) = \operatorname{add}\bigl(\mathsf{S}_1(\mathrm{O},\mathrm{O})\bigr) = \operatorname{add}\bigl(\mathsf{S}_1(\mathrm{B},\mathrm{B})\bigr).$$

**Proof.** It is known that $\mathfrak{u} < \mathfrak{g}$ implies that $\operatorname{add}(\mathcal{N}) = \aleph_1$.[5] By the arguments in the proof of Theorem 4.10 and the results used to prove it (all applying to $\mathsf{S}_1(\mathrm{B},\mathrm{B})$ as well), we have that

$$\operatorname{add}(\mathcal{M}) = \operatorname{cov}(\mathcal{M}) = \operatorname{add}\bigl(\mathsf{S}_1(\mathrm{O},\mathrm{O})\bigr) = \operatorname{add}\bigl(\mathsf{S}_1(\mathrm{B},\mathrm{B})\bigr).$$

Since $\operatorname{add}(\mathcal{M})$ is regular, so is $\operatorname{cov}(\mathcal{M})$. □

**Problem 4.12.** Is it consistent that $\operatorname{add}(\mathcal{N}) < \operatorname{cov}(\mathcal{M}) \leqslant \mathfrak{u} < \mathfrak{g}$?

As in Problem 4.8, we do not know whether $\mathscr{C}_{\operatorname{cov}(\mathcal{M})} = \mathsf{S}_1(\mathrm{O},\mathrm{O})$ in the superperfect set model, or whether there is at all a model where Borel's Conjecture fails (i.e., there are uncountable $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ sets of reals) and $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ is closed under finite products.

## 5. Luzin and Sierpiński sets

Let $\kappa$ be an uncountable cardinal. A set $L \subseteq \mathbb{R}$ is *$\kappa$-Luzin* if its intersection with every meager subset of $\mathbb{R}$ has cardinality less than $\kappa$. *Luzin sets* are $\aleph_1$-Luzin subsets of $\mathbb{R}$. Every $\kappa$-Luzin set $L$ is *$\kappa$-concentrated* on a countable subset $D \subseteq L$ (indeed, on every countable dense subset $D \subseteq L$). A set $S \subseteq \mathbb{R}$ is *$\kappa$-Sierpiński* if its intersection with every Lebesgue null subset of $\mathbb{R}$ has cardinality less than $\kappa$. *Sierpiński sets* are $\aleph_1$-Sierpiński subsets of $\mathbb{R}$.

The starting point of this section, that indeed also led to the earlier two sections, is a surprising result of Babinkostova and Scheepers.[6] Let $L$ be a Luzin set and $S$ be a Sierpiński set. It is known (e.g., [14,29]) that:

(1) $L$ satisfies $\mathsf{S}_1(\mathrm{O},\mathrm{O})$ (indeed, $\mathsf{S}_1(\mathrm{B},\mathrm{B})$), but not $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$.
(2) $S$ satisfies $\mathsf{S}_1(\Gamma,\Gamma)$ (indeed, $\mathsf{S}_1(\mathrm{B}_\Gamma,\mathrm{B}_\Gamma)$), but not $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.

Assuming the Continuum Hypothesis, there is a Luzin set $L$ that does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)$ [14]. It follows that $L \times S$ does not satisfy any of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)$ or $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.

**Remark 5.1.** It follows, in particular, that one cannot improve Theorem 2.2 by proving, e.g., that every concentrated set of real numbers is productively $\mathsf{S}_1(\Gamma,\Gamma)$.

[5] Briefly, if $\operatorname{add}(\mathcal{N}) > \aleph_1$ then there is a rapid filter, but if $\mathfrak{u} < \mathfrak{d}$, in particular if $\mathfrak{u} < \mathfrak{g}$, then no ultrafilter can be rapid, since every ultrafilter is coherent to one with base of cardinality $\mathfrak{u}$, which is smaller than $\mathfrak{d}$.

[6] For the following details, it is recommended to consult the Scheepers Diagram.

**Theorem 5.2** (Babinkostova–Scheepers). *(See [3].) For every Luzin set $L$ and Sierpiński set $S$, $L\times S$ satisfies* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O},\mathrm{O})$.

As mentioned in the earlier sections, Babinkostova and Scheepers prove in [3] that it suffices to assume that $L$ is concentrated on a countable subset (or even less), and that $S$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$. Their full result is generalized further in [37]. In the present section, we settle the question which selective properties are provably satisfied by products of Luzin and Sierpiński sets. First, we use the results of the earlier section to settle the problem in the case of open covers.

**Theorem 5.3.** *For every Luzin set $L$ and Sierpiński set $S$, $L\times S$ satisfies* $\mathsf{S}_1(\Gamma,\mathrm{O})$.

**Proof.** Recall that Luzin sets are concentrated on countable subsets, and Sierpiński sets satisfy $\mathsf{S}_1(\Gamma,\Gamma)$. Apply Theorem 3.5 (or Theorem 2.2). $\square$

With, apparently, no exceptions thus far, all results about selective covering properties of Luzin and Sierpiński sets, proved in the realm of open covers, were also provable for the corresponding Borel-covers variant. Some examples are available in [29]. In light of this, the results in the remainder of this section are surprising. They imply, in particular, that a product of a Luzin and a Sierpiński set may *fail* to satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{B},\mathrm{B})$ (Menger's property for Borel covers), and thus any of the Borel-cover versions of the properties in the Scheepers diagram.

For convenience, in the remainder of this section we work in the Cantor space $\{0,1\}^{\mathbb{N}}$ instead of $\mathbb{R}$. The results can be transformed into $\mathbb{R}$ using the canonical map

$$\begin{aligned}\{0,1\}^{\mathbb{N}}&\to[0,1]\\ f&\mapsto\sum_n\frac{f(n)}{2^n}.\end{aligned}$$

**Definition 5.4.** Define a reflexive binary relation $R$ on $\{0,1\}^{\mathbb{N}}$ by setting $xRy$ if

$$\exists^\infty n,\quad x\restriction[n,2n)=y\restriction[n,2n).$$

For $y\in\{0,1\}^{\mathbb{N}}$, let

$$[y]_R=\bigl\{x\in\{0,1\}^{\mathbb{N}}\colon\ xRy\bigr\}.$$

For $\vec{x}=\langle x_n\colon\ n\in\mathbb{N}\rangle\in(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ and $y\in\{0,1\}^{\mathbb{N}}$, define

$$\mathrm{Match}(\vec{x},y)=\chi_{\{n\colon\ x_nRy\}}.$$

**Lemma 5.5.** *For $x,y\in\{0,1\}^{\mathbb{N}}$:*

(1) *$[y]_R$ is a Lebesgue null, $G_\delta$ dense subset of $\{0,1\}^{\mathbb{N}}$.*
(2) *If $x=^* y$ (equal mod finite), then $[x]_R=[y]_R$.*
(3) $\mathrm{Match}:(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}\times\{0,1\}^{\mathbb{N}}\to\{0,1\}^{\mathbb{N}}$ *is a Borel map.* $\square$

**Theorem 5.6.** *For all comeager $X\subseteq(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ and nonnull $Y\subseteq\{0,1\}^{\mathbb{N}}$,* $\mathrm{Match}[X\times Y]=\{0,1\}^{\mathbb{N}}$.

**Proof.**

**Lemma 5.7.** *Let $Y \subseteq \{0,1\}^{\mathbb{N}}$ be nonnull. For each $\vec{x} \in (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$, there is $y \in Y$ such that $\vec{x} \in ([y]_R{}^{\mathsf{c}})^{\mathbb{N}}$.*

**Proof.** By Lemma 5.5, $[x_n]_R$ is null for all $n$, and consequently so is $\bigcup_n [x_n]_R$. Pick

$$y \in Y \setminus \bigcup_n [x_n]_R.$$

As $R$ is symmetric, $\vec{x} \in ([y]_R{}^{\mathsf{c}})^{\mathbb{N}}$. $\square$

**Lemma 5.8.** *Let $Y \subseteq \{0,1\}^{\mathbb{N}}$ be nonnull. For each $I \subseteq \mathbb{N}$, $\bigcup_{y\in Y} ([y]_R{}^{\mathsf{c}})^{I^{\mathsf{c}}} \times ([y]_R)^I$ is nonmeager in $(\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$.*

**Proof.** Let $\vec{x} \in (\{0,1\}^{\mathbb{N}})^{I^{\mathsf{c}}}$. By Lemma 5.7, there is $y \in Y$ such that $\vec{x} \in ([y]_R{}^{\mathsf{c}})^{I^{\mathsf{c}}}$. By Lemma 5.5, $([y]_R)^I$ is comeager in $(\{0,1\}^{\mathbb{N}})^I$. Since

$$\{\vec{x}\} \times \big([y]_R\big)^I \subseteq \bigcup_{y\in Y} \big([y]_R{}^{\mathsf{c}}\big)^{I^{\mathsf{c}}} \times \big([y]_R\big)^I,$$

it follows that all vertical sections of $\bigcup_{y\in Y} ([y]_R{}^{\mathsf{c}})^{I^{\mathsf{c}}} \times ([y]_R)^I$ are comeager, in particular nonmeager. By [15, Lemma 8.42], our set is nonmeager in $(\{0,1\}^{\mathbb{N}})^{I^{\mathsf{c}}} \times (\{0,1\}^{\mathbb{N}})^I$. $\square$

Let $z = \chi_I \in \{0,1\}^{\mathbb{N}}$. By Lemma 5.8, there is

$$\vec{x} \in X \cap \left( \bigcup_{y\in Y} \big([y]_R{}^{\mathsf{c}}\big)^{I^{\mathsf{c}}} \times \big([y]_R\big)^I \right).$$

Then $\mathrm{Match}(\vec{x}, y) = \chi_I = z$. $\square$

**Corollary 5.9.** *There is a Borel map $f : \{0,1\}^{\mathbb{N}} \to \{0,1\}^{\mathbb{N}}$ such that, for all comeager $X \subseteq \{0,1\}^{\mathbb{N}}$ and nonnull $Y \subseteq \{0,1\}^{\mathbb{N}}$, $f[X \times Y] = \{0,1\}^{\mathbb{N}}$.*

**Proof.** The canonical bijection $\{0,1\}^{\mathbb{N}} \to (\{0,1\}^{\mathbb{N}})^{\mathbb{N}}$ is Borel, and preserves meager and null sets in both directions. $\square$

Sierpiński sets are special kinds of nonnull sets. In the Sacks model, there are Luzin and Sierpiński sets, but they are all of cardinality $\aleph_1$, whereas the continuum is $\aleph_2$. Thus, consistently, there are no Luzin and Sierpiński sets whose product can be mapped onto $\{0,1\}^{\mathbb{N}}$. However, we have the following.

**Corollary 5.10** (CH). *For each nonnull set $Y \subseteq \{0,1\}^{\mathbb{N}}$, there is a Luzin set $L \subseteq \{0,1\}^{\mathbb{N}}$ such that $\{0,1\}^{\mathbb{N}}$ is a Borel image of $L \times Y$.*

**Proof.** Let $f$ be the function defined in Corollary 5.9. Enumerate $\{0,1\}^{\mathbb{N}} = \{r_\alpha\colon \alpha < \aleph_1\}$. Let $\{M_\alpha\colon \alpha < \aleph_1\}$ be a cofinal family of meager subsets of $\{0,1\}^{\mathbb{N}}$. For each $\alpha < \aleph_1$,

$$f\left[\left(\{0,1\}^{\mathbb{N}} \setminus \bigcup_{\beta<\alpha} M_\alpha\right) \times Y\right] = \{0,1\}^{\mathbb{N}}.$$

Pick $(x_\alpha, y_\alpha) \in (\{0,1\}^{\mathbb{N}} \setminus \bigcup_{\beta<\alpha} M_\alpha) \times Y$ such that $f(x_\alpha, y_\alpha) = r_\alpha$. Finally, let $L = \{x_\alpha\colon \alpha < \aleph_1\}$. $\square$

$\mathrm{add}(\mathcal{M})$-Luzin sets satisfy $\mathsf{S}_1(\mathrm{B},\mathrm{B})$ (e.g., [29]).

**Corollary 5.11** ($\mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$). *For each nonnull set $Y \subseteq \{0,1\}^{\mathbb{N}}$, there is an $\mathrm{add}(\mathcal{M})$-Luzin set $L \subseteq \{0,1\}^{\mathbb{N}}$ such that $L \times Y$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{B},\mathrm{B})$.*

**Proof.** Let $f$ be the function defined in Corollary 5.9. Let $\kappa = \mathrm{add}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$. As $\mathrm{add}(\mathcal{M}) \leqslant \mathfrak{d} \leqslant \mathrm{cof}(\mathcal{M})$ (in ZFC), there is a dominating set $\{d_\alpha\colon \alpha < \kappa\} \subseteq \mathbb{N}^{\mathbb{N}}$. Identify $\{0,1\}^{\mathbb{N}}$ with $\mathbb{N}^{\mathbb{N}}$ via a Borel bijection.

Let $\{M_\alpha\colon \alpha < \kappa\}$ be a cofinal family of meager subsets of $\{0,1\}^{\mathbb{N}}$. For each $\alpha < \aleph_1$,

$$f\left[\left(\{0,1\}^{\mathbb{N}} \setminus \bigcup_{\beta<\alpha} M_\alpha\right) \times Y\right] = \mathbb{N}^{\mathbb{N}}.$$

Pick $(x_\alpha, y_\alpha) \in (\{0,1\}^{\mathbb{N}} \setminus \bigcup_{\beta<\alpha} M_\alpha) \times Y$ such that $f(x_\alpha, y_\alpha) = d_\alpha$. Finally, let $L = \{x_\alpha\colon \alpha < \aleph_1\}$.

As the Borel image $f[L \times Y]$ contains $D$, it is dominating. Thus, $L \times Y$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{B},\mathrm{B})$. □

**Theorem 5.12** ($\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$). *There are an $\mathrm{add}(\mathcal{N})$-Luzin set $L \subseteq \{0,1\}^{\mathbb{N}}$ and an $\mathrm{add}(\mathcal{N})$-Sierpiński set $S \subseteq \{0,1\}^{\mathbb{N}}$ such that:*

(1) *All finite powers of $L$ satisfy $\mathsf{S}_1(\mathrm{B}_\Omega,\mathrm{B}_\Omega)$;*
(2) *All finite powers of $S$ satisfy $\mathsf{S}_1(\mathrm{B}_\Gamma,\mathrm{B}_\Gamma)$ and $\mathsf{S}_{\mathrm{fin}}(\mathrm{B}_\Omega,\mathrm{B}_\Omega)$; but*
(3) *$L \times S$ does not satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{B},\mathrm{B})$.*

*Moreover, $L$ does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Gamma)$ and $S$ does not satisfy $\mathsf{S}_1(\mathrm{O},\mathrm{O})$.*

**Proof.** As $\mathrm{add}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$, there are an $\mathrm{add}(\mathcal{N})$-Sierpiński set $S$ as in (2) [32, Corollary 25], and by a dual argument, an $\mathrm{add}(\mathcal{N})$-Luzin set $L$ such that all finite powers of $L$ satisfy $\mathsf{S}_1(\mathrm{B},\mathrm{B})$. (Here, we use Carlson's Theorem, that the union of less than $\mathrm{add}(\mathcal{N})$ elements of $\mathsf{S}_1(\mathrm{B},\mathrm{B})$ is in $\mathsf{S}_1(\mathrm{B},\mathrm{B})$ [33].) It is pointed out in [29, Theorem 18] that if all finite powers of $X$ have property $\mathsf{S}_1(\mathrm{B},\mathrm{B})$, then $X$ has property $\mathsf{S}_1(\mathrm{B}_\Omega,\mathrm{B}_\Omega)$. This implies (1).

During the construction of $L$, one can also accommodate the restrictions provided in the proof of Corollary 5.11, to make sure that $f[L \times S]$ contains a (Borel preimage in $\{0,1\}^{\mathbb{N}}$ of $a$) dominating subset of $\mathbb{N}^{\mathbb{N}}$. This gives (3).

The last assertion in the theorem is due to Sierpiński, cf. [14]. □

## 6. Scales and $\mathfrak{b}$-scales

In the earlier sections, we have discussed Luzin sets as special examples of concentrated sets. Another standard method for constructing concentrated sets, initiated by Rothberger, is that of using *scales*. These constructions require in general milder hypotheses than those used for the construction of Luzin and Sierpiński sets, and in many cases can be carried out outright in ZFC.

For our purposes, it is convenient to identify the Cantor space $\{0,1\}^{\mathbb{N}}$ with $P(\mathbb{N}) = [\mathbb{N}]^\infty \cup [\mathbb{N}]^{<\infty}$, and use the induced topology. For $a \in [\mathbb{N}]^\infty$ and $n \in \mathbb{N}$, $a(n)$ denotes the $n$-th element in the increasing enumeration of $a$. For $a, b \in [\mathbb{N}]^\infty$, $a \leqslant^* b$ means that $a(n) \leqslant b(n)$ for all but finitely many $n$. A *scale* is a cofinal (dominating) set $S = \{s_\alpha\colon \alpha < \kappa\}$ in $([\mathbb{N}]^\infty, \leqslant^*)$ such that $s_\alpha \leqslant^* s_\beta$ for $\alpha < \beta$. Scales exist if and only if $\mathfrak{b} = \mathfrak{d}$, and in this case, their cardinality is $\mathfrak{b}$. If we generalize "cofinal" to "unbounded", we obtain the definition of $\mathfrak{b}$-*scale*, an object constructible within ZFC. For each $\mathfrak{b}$-scale $B$, $B \cup [\mathbb{N}]^{<\infty}$ is $\mathfrak{b}$-concentrated on its countable subset $[\mathbb{N}]^{<\infty}$.

For brevity, the union of a scale with $[\mathbb{N}]^{<\infty}$, viewed as a subset of the Cantor space $P(\mathbb{N})$, will be called *scale set*. $\mathfrak{b}$-*scale sets* are defined similarly.

**Corollary 6.1.** *Every product of a $\mathfrak{b}$-scale set and a $\mathrm{cov}(\mathcal{M})$-concentrated space satisfies $\mathsf{S}_1(\Gamma, \mathrm{O})$.*

**Proof.** Every $\mathfrak{b}$-scale set satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ and, being $\mathfrak{d}$-concentrated, $\mathsf{S}_1(\Gamma, \mathrm{O})$ as well [7]. Apply Proposition 3.2 and Theorem 3.5. $\square$

**Theorem 6.2.** *Let $S \subseteq [\mathbb{N}]^\infty$ be a scale. The scale set $S \cup [\mathbb{N}]^{<\infty}$ is productively $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ and productively $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega)$.*

**Proof.** Since $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega) = \mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})^\uparrow$, it suffices to prove the first assertion. We use the projection method (cf. Theorem 2.2).

Let $Y$ be a space satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Let $\mathcal{U}_n$, $n \in \mathbb{N}$, be covers of $(S \cup [\mathbb{N}]^{<\infty}) \times Y$ by sets open in $P(\mathbb{N}) \times Y$. As $[\mathbb{N}]^{<\infty}$ is countable, $[\mathbb{N}]^{<\infty} \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Pick finite sets $\mathcal{F}_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $[\mathbb{N}]^{<\infty} \times Y \subseteq U := \bigcup_n \bigcup \mathcal{F}_n$.

Since $Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ and $P(\mathbb{N})$ is compact, the projection $M$ of $(P(\mathbb{N}) \times Y) \setminus U$ on the first coordinate satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. $M \subseteq [\mathbb{N}]^\infty$, and satisfying $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, it is not dominating. Thus, $|M \cap S| < \mathfrak{d} = \mathfrak{b}$. It follows that $(M \cap S) \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Pick finite sets $\mathcal{F}'_n \subseteq \mathcal{U}_n$, $n \in \mathbb{N}$, such that $(M \cap S) \times Y \subseteq \bigcup_n \bigcup \mathcal{F}'_n$. Then

$$X \times Y \subseteq \bigcup_n \bigcup (\mathcal{F}_n \cup \mathcal{F}'_n). \qquad \square$$

Theorem 6.2 is the last one in this paper proved by the projection method. In order to establish additional productive properties of scale sets, we use the following method.

**Lemma 6.3** (Productive Two Worlds Lemma). *Let $Y$ be a space, and for each $n$, let*

$$\{U^n_m \colon m \in \mathbb{N}\} \in \Omega\big([\mathbb{N}]^{<\infty} \times Y\big)$$

*with each $U^n_m$ clopen in $P(\mathbb{N}) \times Y$. There is a continuous map $\Psi : Y \to \mathbb{N}^{\mathbb{N}}$ such that, for each $n$,*

$$(x, y) \in U^n_{\leqslant \Psi(y)(n)}$$

*for all $x \in [\mathbb{N}]^\infty$ such that $|x| < n$ or $\Psi(y)(n) \leqslant x(n)$.*

**Proof.** Let $y \in Y$. Fix $n$. Let $m_n(1) = a_n(1) = 1$. By induction on $k$, let $m_n(k+1)$ be minimal with

$$P\big(\{1, \ldots, a_n(k) - 1\}\big) \times \{y\} \subseteq U^n_{m_n(k+1)},$$

and let $a_n(k+1)$ be minimal such that

$$(x, y) \in U^n_{m_n(k+1)}$$

for all $x \in P(\mathbb{N})$ with $x \cap \{a_n(k), \ldots, a_n(k+1) - 1\} = \emptyset$.

Define

$$\Psi(y)(n) = \max\big\{a_n(n+1), m_n(n+1)\big\}$$

for all $n$. $\Psi$ is continuous. Fix $n$, and let $x \in [\mathbb{N}]^\infty$ with $\Psi(y)(n) \leqslant x(n)$. As $a_n(n+1) \leqslant \Psi(y)(n) \leqslant x(n)$, there is $k < n+1$ with $x \cap \{a_n(k), \ldots, a_n(k+1) - 1\} = \emptyset$. Then $(x, y) \in U^n_{m_n(k+1)}$. As

$$m_n(k+1) \leqslant m_n(n+1) \leqslant \Psi(y)(n),$$

$U^n_{m_n(k+1)} \subseteq U^n_{\leqslant\Psi(y)(n)}$. $\square$

**Lemma 6.4.** *Let $X$ be a space and $\mathscr{A} \in \{\Gamma, \Omega, \mathrm{O}\}$. If there is a space $Y$ such that $X \times Y$ is Lindelöf but not $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathscr{A})$, then there is such a subspace $Y$ of the Cantor space.*

**Proof.** Let $\mathcal{U}_n$, $n \in \mathbb{N}$, be open covers of $X \times Y$ witnessing the failure of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathscr{A})$. As $X \times Y$ is Lindelöf, we may assume that each $\mathcal{U}_n$ has the form $\{U^n_m \times V^n_m\colon m \in \mathbb{N}\}$. Define set-valued maps $\Phi, \Psi$ from $X, Y$, respectively, into the Cantor space $P(\mathbb{N} \times \mathbb{N})$ by

$$\begin{aligned} A \in \Phi(x) \quad &\Longleftrightarrow \quad \bigl\{(n,m)\colon x \in U^n_m\bigr\} \subseteq A, \\ A \in \Psi(y) \quad &\Longleftrightarrow \quad \bigl\{(n,m)\colon y \in V^n_m\bigr\} \subseteq A \end{aligned}$$

for all $x \in X, y \in Y$. By [38, Lemma 2], these maps are compact-valued, upper semicontinuous. Thus,

$$\Psi[Y] := \bigcup_{y \in Y} \Psi(y) \subseteq P(\mathbb{N} \times \mathbb{N})$$

satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathscr{A})$, and $X \times \Psi[Y]$ is Lindelöf, being a compact-valued, upper semicontinuous image of the Lindelöf space $X \times Y$.

We claim that $X \times \Psi[Y]$ does not satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathscr{A})$. Assume otherwise. Then $\Phi[X] \times \Psi[Y]$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \mathscr{A})$, being a compact-valued, upper semicontinuous image of $X \times \Psi[Y]$. Define

$$\begin{aligned} \Xi : P(\mathbb{N} \times \mathbb{N}) \times P(\mathbb{N} \times \mathbb{N}) &\to P(\mathbb{N} \times \mathbb{N}) \\ (A, B) &\mapsto A \cap B. \end{aligned}$$

Then $\Xi$ is continuous. For each $n$, let $\mathcal{W}_n := \{W^n_m\colon m \in \mathbb{N}\}$, where

$$W^n_m = \bigl\{A \subseteq \mathbb{N} \times \mathbb{N}\colon (n,m) \in A\bigr\}.$$

For each $n$, $\mathcal{W}_n$ is an open cover of $\Xi(\Phi[X] \times \Psi[Y])$. But $\{W^n_{\leqslant f(n)}\colon n \in \mathbb{N}\}$ is not in $\mathscr{A}(\Xi(\Phi[X] \times \Psi[Y]))$ for any $f \in \mathbb{N}^{\mathbb{N}}$, for otherwise, $\{\bigcup_{m \leqslant f(n)} U^n_m \times V^n_m\colon n \in \mathbb{N}\}$ would be in $\mathscr{A}(X \times Y)$. $\square$

An open cover $\mathcal{U} \in \Omega^{\mathrm{gp}}(X)$ if there are $h \in \mathbb{N}^{\mathbb{N}}$ and an enumeration $\mathcal{U} = \{U_n\colon n \in \mathbb{N}\}$ such that, for each finite $F \subseteq X$ and each $n$, $F \subseteq U_k$ for some $h(n) \leqslant k \leqslant h(n+1)$. $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)^{\uparrow} = \mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$ [17].

Bartoszyński and Shelah [6] proved that every $\mathfrak{b}$-scale set satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. Then, Bartoszyński and Tsaban [7] proved that all finite powers of a $\mathfrak{b}$-scale set satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. Later, Tsaban and Zdomskyy [35] proved that all finite products of $\mathfrak{b}$-scale sets satisfy $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. The following theorem is much stronger.

**Theorem 6.5.** *Every $\mathfrak{b}$-scale set (in particular, every scale set) is:*

(1) *Productively hereditarily Lindelöf* $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$; *and*
(2) *Productively hereditarily Lindelöf* $\mathsf{S}_{\mathrm{fin}}(\Omega, \Omega^{\mathrm{gp}})$.

**Proof.** It suffices to prove the first assertion.

Let $B = \{b_\alpha\colon \alpha < \mathfrak{b}\}$ be a $\mathfrak{b}$-scale. Let $Y$ be a hereditarily Lindelöf space satisfying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. Then $(B \cup [\mathbb{N}]^{\infty}) \times Y$ is (hereditarily) Lindelöf. By Lemma 6.4, we may assume that $Y \subseteq \{0,1\}^{\mathbb{N}}$.

Let $\mathcal{U}_n \in \Omega((B \cup [\mathbb{N}]^{<\infty}) \times Y)$ for all $n$. As $X \times Y$ is a subspace of the Cantor space, we may assume that each element of each $\mathcal{U}_n$ is clopen, and the Productive Two Worlds Lemma applies. Let $\Psi$ be as in that lemma. As $Y$ satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$, $\Psi[Y]$ is bounded by some $g \in \mathbb{N}^{\mathbb{N}}$. Take $\alpha < \mathfrak{b}$ such that the set $I = \{n\colon\ g(n) \leqslant b_\alpha(n)\}$ is infinite. For each $\beta \geqslant \alpha$ and each $y \in Y$,

$$\Psi(y)(n) \leqslant g(n) \leqslant b_\alpha(n) \leqslant b_\beta(n),$$

and therefore $(b_\beta, y) \in U^n_{\leqslant g(n)}$, for all but finitely many $n \in I$. We also have that $(x, y) \in U^n_{\leqslant g(n)}$, for all $x \in [\mathbb{N}]^{<\infty}$, $y \in Y$ and all but finitely many $n$.

As $\operatorname{add}(\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)) = \mathfrak{b}$, $\{b_\beta\colon\ \beta < \alpha\} \times Y$ satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$, and thus there is $h \in \mathbb{N}^{\mathbb{N}}$ such that $\{U^n_{\leqslant h(n)}\colon\ n \in \mathbb{N}\} \in \Gamma(\{b_\beta\colon\ \beta < \alpha\} \times Y)$. For $n \in I$, let

$$\mathcal{F}_n = \bigl\{U^n_m\colon\ m \leqslant \max\bigl\{g(n), h(n)\bigr\}\bigr\}.$$

For $n \notin I$, let $\mathcal{F}_n = \emptyset$. Then $\{\bigcup \mathcal{F}_n\colon\ n \in \mathbb{N}\} \in \Gamma((B \cup [\mathbb{N}]^{<\infty}) \times Y)$. $\square$

A set $D \subseteq \mathbb{N}^{\mathbb{N}}$ is *finitely dominating* if its closure under pointwise maxima of finite subsets, $\operatorname{maxfin}(D)$, is dominating. Let $\operatorname{cov}(\mathfrak{D}_{\text{fin}})$ be the minimal $\kappa$ such that $\mathbb{N}^{\mathbb{N}}$ (equivalently, a dominating subset of $\mathbb{N}^{\mathbb{N}}$) can be decomposed into $\kappa$ many sets, none of which finitely dominating. Then

$$\max\{\mathfrak{b}, \mathfrak{g}\} \leqslant \operatorname{cov}(\mathfrak{D}_{\text{fin}}) \leqslant \mathfrak{d},$$

and strict inequalities are consistent [19].

**Lemma 6.6.** *Every space of cardinality less than* $\operatorname{cov}(\mathfrak{D}_{\text{fin}})$ *is productively* $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$ *for countable covers.*

**Proof.** Assume that $|X| < \operatorname{cov}(\mathfrak{D}_{\text{fin}})$ and $Y$ satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$ for countable covers. Using the terminology of the forthcoming Section 7 and Theorem 7.4, let $\Psi : X \times Y \to \mathbb{N}^{\mathbb{N}}$ be upper continuous. It suffices to prove that $\Psi[X \times Y]$ is not finitely dominating. For each finite $F \subseteq X$, the map

$$\begin{aligned} \Psi_F : Y &\to \mathbb{N}^{\mathbb{N}} \\ y &\mapsto \max\bigl\{\Psi(x, y)\colon\ x \in F\bigr\} \end{aligned}$$

is upper continuous. Thus, $\Psi_F[Y]$ is not finitely dominating. Each finite subset of $X \times Y$ is contained in one of the form $F_1 \times F_2$, and

$$\max \Psi[F_1 \times F_2] = \max \Psi_{F_1}[F_2].$$

Thus,

$$\bigcup_{F \in [X]^{<\infty}} \operatorname{maxfin} \Psi_F[Y]$$

is cofinal in $\operatorname{maxfin} \Psi[X \times Y]$. This is a directed union (every finite sub-union is contained in a single member) of less than $\operatorname{cov}(\mathfrak{D}_{\text{fin}})$ many sets that are not finitely dominating. Thus, it is not finitely dominating. $\square$

**Theorem 6.7.** *For each scale* $S \subseteq [\mathbb{N}]^\infty$*, the scale set* $S \cup [\mathbb{N}]^{<\infty}$ *is productively hereditarily Lindelöf* $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$.

**Proof.** Let $S = \{s_\alpha\colon\ \alpha < \mathfrak{b}\}$ be a scale, and let $Y$ be a space satisfying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)$. By Lemma 6.4, we may assume that $Y$ is a subspace of the Cantor space. Let

$$\mathcal{U}_n = \{U^n_m\colon\ m \in \mathbb{N}\} \in \Omega\big(\big(S \cup [\mathbb{N}]^{<\infty}\big) \times Y\big)$$

for all $n$. We may assume that every $U^n_m$, $n, m \in \mathbb{N}$, is clopen. Let $\Psi$ be as in Lemma 6.3. For $\alpha < \gamma < \mathfrak{b}$, let

$$X_{\alpha,\gamma} = \{s_\beta\colon\ \beta < \alpha \text{ or } \gamma \leqslant \beta\} \cup [\mathbb{N}]^{<\infty}.$$

**Lemma 6.8.** *For each $\alpha < \mathfrak{b}$, there are $\gamma < \mathfrak{b}$ and $g \in \mathbb{N}^{\mathbb{N}}$ such that*

$$\{U^n_{\leqslant g(n)}\colon\ n \in \mathbb{N}\} \in \Omega(X_{\alpha,\gamma} \times Y).$$

**Proof.** Fix $p \notin \{s_\beta\colon\ \beta < \alpha\}$, and consider $D = \{s_\beta\colon\ \beta < \alpha\} \cup \{p\}$ as a discrete space. By Lemma 6.6, $D \times Y$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O},\Omega)$. Define $\Phi : D \times Y \to \mathbb{N}^{\mathbb{N}}$ by

$$\begin{aligned}
&\Phi(p, y) = \Psi(y);\\
&\Phi(s_\beta, y)(n) = \min\{m\colon\ (s_\beta, y) \in U^n_m\} \quad (n \in \mathbb{N}).
\end{aligned}$$

As $\Phi$ is continuous, $\Phi[D \times Y]$ is not finitely dominating. Let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness for that. Let $\gamma$ be such that $g \leqslant^* s_\gamma$. We claim that $\gamma$ and $g$ are as required. Let $F \subseteq X_{\alpha,\gamma}$ and $G \subseteq Y$ be finite sets. Decompose $F$ as

$$F = \big(F \cap X_{\alpha,\mathfrak{b}} \setminus [\mathbb{N}]^{<\infty}\big) \cup \big(F \cap [\mathbb{N}]^{<\infty}\big) \cup \big(F \cap X_{0,\gamma} \setminus [\mathbb{N}]^{<\infty}\big).$$

The set

$$I := \big\{n\colon\ \max \Phi\big[\big(\big(F \cap X_{\alpha,\mathfrak{b}} \setminus [\mathbb{N}]^{<\infty}\big) \cup \{p\}\big) \times G\big](n) \leqslant g(n)\big\}$$

is infinite.

For each $s_\beta \in F \cap X_{\alpha,\mathfrak{b}} \setminus [\mathbb{N}]^{<\infty}$ and each $y \in Y$,

$$\Phi(s_\beta, y)(n) \leqslant g(n)$$

for all $n \in I$. Thus, $(F \cap X_{\alpha,\mathfrak{b}} \setminus [\mathbb{N}]^{<\infty}) \times G \subseteq U^n_{\leqslant g(n)}$ for all $n \in I$.

For each $x \in [\mathbb{N}]^{<\infty}$ and each $y \in G$,

$$\Psi(y)(n) = \Phi(p, y)(n) \leqslant g(n) \quad \text{and} \quad |x| < n$$

for all but finitely many $n \in I$. Thus, $(F \cap [\mathbb{N}]^{<\infty}) \times G \subseteq U^n_{\leqslant g(n)}$ for all but finitely many $n \in I$.

Finally, for each $\beta \geqslant \gamma$ and each $y \in G$,

$$\Psi(y)(n) = \Phi(p, y)(n) \leqslant g(n) \leqslant s_\beta(n)$$

for all but finitely many $n \in I$. Thus, $(F \cap X_{0,\gamma} \setminus [\mathbb{N}]^{<\infty}) \times G \subseteq U^n_{\leqslant g(n)}$ for all but finitely many $n \in I$.

It follows that $F \times G \subseteq U^n_{\leqslant g(n)}$ for all but finitely many $n \in I$. $\square$

By Lemma 6.8 applied to $\alpha = 0$, there are $\gamma_1 < \mathfrak{b}$ and $g_1 \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{U^n_{\leqslant g_1(n)}\colon\ n \in \mathbb{N}\} \in \Omega(X_{0,\gamma_1} \times Y).$$

By Lemma 6.8 applied to $\alpha = \gamma_1$, there are $\gamma_2 < \mathfrak{b}$ and $g_2 \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{U^n_{\leqslant g_2(n)}:\ n \in \mathbb{N}\} \in \Omega(X_{\gamma_1,\gamma_2} \times Y).$$

By Lemma 6.8 applied to $\alpha = \gamma_2$, there are $\gamma_3 < \mathfrak{b}$ and $g_3 \in \mathbb{N}^{\mathbb{N}}$ such that

$$\{U^n_{\leqslant g_3(n)}:\ n \in \mathbb{N}\} \in \Omega(X_{\gamma_2,\gamma_3} \times Y).$$

Continue, in this manner, to define $\gamma_n$ and $g_n$ for all $n$. Let $\gamma = \sup_n \gamma_n$, and $g$ be a $\leqslant^*$-bound of $\{g_n:\ n \in \mathbb{N}\}$. Then

$$\{U^n_{\leqslant g(n)}:\ n \in \mathbb{N}\} \in \Omega\bigl(\bigl(S \cup [\mathbb{N}]^{<\infty}\bigr) \times Y\bigr).$$

Indeed, let $F \times G$ be a finite subset of $(S \cup [\mathbb{N}]^{<\infty}) \times Y$. Let $H = \{\beta < \mathfrak{b}:\ s_\beta \in F\}$. As $H$ is finite, there is $k$ such that $H \cap \gamma \subseteq \gamma_k$. Then $F \subseteq X_{\gamma_k,\gamma_{k+1}}$, and thus there are infinitely many $n$ such that

$$F \times G \subseteq U^n_{\leqslant g_k(n)} \subseteq U^n_{\leqslant g(n)}. \qquad \square$$

We do not know whether the hypothesis in the following corollary is necessary.

**Corollary 6.9** ($\mathfrak{b} \leqslant \mathrm{cov}(\mathcal{M})$). *Every $\mathfrak{b}$-scale set is:*

(1) *Productively hereditarily Lindelöf* $\mathsf{S}_1(\Omega, \daleth(\Gamma))$;
(2) *Productively hereditarily Lindelöf* $\mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}})$;
(3) *Productively hereditarily Lindelöf* $\mathsf{S}_1(\Gamma, \daleth(\Gamma))$; *and*
(4) *Productively hereditarily Lindelöf* $\mathsf{S}_1(\Gamma, \daleth(\Gamma))^{\uparrow}$.

**Proof.** The second assertion follows from the first, since $\mathsf{S}_1(\Omega, \Omega^{\mathrm{gp}}) = \mathsf{S}_1(\Omega, \daleth(\Gamma))^{\uparrow}$ [17]. Similarly, the fourth assertion follows from the third.

(1) Let $X$ be a $\mathfrak{b}$-scale set, and let $Y$ be hereditarily Lindelöf $\mathsf{S}_1(\Omega, \daleth(\Gamma))$. As $X$ is $\mathfrak{b}$-concentrated, it is in particular $\mathrm{cov}(\mathcal{M})$-concentrated. By [37, Theorem 3.1(2)], $X \times Y$ satisfies $\mathsf{S}_1(\mathrm{O}, \mathrm{O})$. By Theorem 6.5, $X \times Y$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$. To conclude, recall that $\mathsf{S}_1(\Omega, \daleth(\Gamma)) = \mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma) \cap \mathsf{S}_1(\mathrm{O}, \mathrm{O})$ [17].

(3) Similar, using Proposition 3.2, Theorem 3.5 and Theorem 6.5. $\square$

**Remark 6.10.** The only role of our restriction to hereditarily Lindelöf in the results of this section is to guarantee that the product with a scale set remains Lindelöf.

## 7. Combinatorial characterizations of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Omega)$, and $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$

We provide here characterizations of $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Omega)$, and $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ for arbitrary topological spaces. These characterizations will be used in the following section. In this section only, the spaces are not assumed to be Tychonoff, so that the characterizations may find additional future applications in more general contexts. Replacing *upper continuous* by *continuous* and restricting attention to separable, metrizable, zero-dimensional spaces, the first two items in each of our characterizations become the celebrated characterizations of Hurewicz–Recław [22] (cf. [30]).

**Definition 7.1.** Let $X$ be a topological space. For each $m$ and $n$, consider the basic open set

$$O^n_m = \{f \in \mathbb{N}^{\mathbb{N}}:\ f(n) \leqslant m\} = \pi_n^{-1}\bigl[\{1, \dots, m\}\bigr]$$

in $\mathbb{N}^{\mathbb{N}}$, where $\pi_n : \mathbb{N}^{\mathbb{N}} \to \mathbb{N}$ is the projection on the $n$-th coordinate. A function $\Psi : X \to \mathbb{N}^{\mathbb{N}}$ is *upper continuous* if, for each $n$, the set

$$\Psi^{-1}\big[O^n_m\big] = \big\{x \in X\colon\ \Psi(x)(n) \leqslant m\big\}$$

is open in $X$.

A set-valued map $\Psi$ from $X$ to $\mathbb{N}^{\mathbb{N}}$ is *principal* if there is a function $\psi : X \to \mathbb{N}^{\mathbb{N}}$ such that

$$\Psi(x) = \big\{f \in \mathbb{N}^{\mathbb{N}}\colon\ \forall n,\ f(n) \leqslant \psi(x)(n)\big\}.$$

We use *cusco* as abbreviation for *compact-valued upper semicontinuous.*

**Lemma 7.2.** *Let $X$ be a topological space.*

(1) *Every cusco map from $X$ to $\mathbb{N}^{\mathbb{N}}$ is dominated by a principal one.*
(2) *A function $\psi : X \to \mathbb{N}^{\mathbb{N}}$ is upper continuous if and only if the principal set-valued map $\Psi$ from $X$ to $\mathbb{N}^{\mathbb{N}}$ determined by $\psi$ is cusco.* $\square$

The equivalence of (1) and (4) in the following theorem is established in [38, Theorem 8]. Our proof is, perhaps, more transparent.

**Theorem 7.3.** *Let $X$ be a topological space. The following assertions are equivalent*:

(1) *$X$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$;*
(2) *$X$ is Lindelöf, and every upper continuous image of $X$ in $\mathbb{N}^{\mathbb{N}}$ is bounded*;
(3) *$X$ is Lindelöf, and every principal cusco image of $X$ in $\mathbb{N}^{\mathbb{N}}$ is bounded*;
(4) *$X$ is Lindelöf, and every cusco image of $X$ in $\mathbb{N}^{\mathbb{N}}$ is bounded.*

**Proof.** The equivalence of (2), (3), (4) follows from Lemma 7.2.

(1) $\Rightarrow$ (2): Let $\Psi : X \to \mathbb{N}^{\mathbb{N}}$ be upper continuous. For each $n$ and each $m$, let

$$U^n_m = \big\{x \in X\colon\ \Psi(x)(n) \leqslant m\big\}.$$

The sets $U^n_m$ increase with $m$, and $\{U^n_m\colon\ m \in \mathbb{N}\}$ is an open cover of $X$. We may assume that $U^n_m \neq X$ for all $m$. (Otherwise, treat the indices $m$ with $U^n_m = X$.)

Applying $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, there are $m_1, m_2, \ldots$ such that $\{U^n_{m_n}\colon\ n \in \mathbb{N}\} \in \Gamma(X)$. For each $x \in X$, $x \in U^n_{m_n}$, and thus $\Psi(x)(n) \leqslant m_n$, for all but finitely many $n$. In other words, $\Psi[X]$ is bounded by the function $g(n) = m_n$.

(2) $\Rightarrow$ (1): For each $n$, let $\mathcal{U}_n = \{U^n_m\colon\ m \in \mathbb{N}\}$ be an open cover of $X$. We may assume that the sets $U^n_m$ increase with $m$. For each $x \in X$, define

$$\Psi(x)(n) = \min\big\{m\colon\ x \in U^n_m\big\}$$

for all $n$. Then $\Psi$ is upper continuous. Indeed, for each $n$,

$$\Psi^{-1}\big[\{1,\ldots,m\}\big] = \Psi^{-1}\big[\{1\}\big] \cup \cdots \cup \Psi^{-1}\big[\{m\}\big] = U^n_1 \cup \cdots \cup U^n_m = U^n_m$$

is open in $X$.

Let $g \in \mathbb{N}^{\mathbb{N}}$ be a bound of $\Psi[X]$. Then $\{U^n_{g(n)}\colon n \in \mathbb{N}\} \in \Gamma(X)$. Indeed, for each $x \in X$, $\Psi(x)(n) \leqslant g(n)$, and thus $x \in U^n_{g(n)}$, for all but finitely many $n$. $\square$

Similarly, we have the following.

**Theorem 7.4.** *Let $X$ be a topological space. The following assertions are equivalent*:

(1) *$X$ satisfies $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$*;
(2) *$X$ is Lindelöf, and no upper continuous image of $X$ in $\mathbb{N}^{\mathbb{N}}$ is finitely dominating*;
(3) *$X$ is Lindelöf, and no principal cusco image of $X$ in $\mathbb{N}^{\mathbb{N}}$ is finitely dominating;*
(4) *$X$ is Lindelöf, and no cusco image of $X$ in $\mathbb{N}^{\mathbb{N}}$ is finitely dominating.* $\square$

**Theorem 7.5.** *Let $X$ be a topological space. The following assertions are equivalent*:

(1) *$X$ satisfies $\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$*;
(2) *$X$ is Lindelöf, and no upper continuous image of $X$ in $\mathbb{N}^{\mathbb{N}}$ is dominating*;
(3) *$X$ is Lindelöf, and $\mathbb{N}^{\mathbb{N}}$ is not a principal cusco image of $X$*;
(4) *$X$ is Lindelöf, and $\mathbb{N}^{\mathbb{N}}$ is not a principal cusco image of $X$.* $\square$

While cusco images preserve the properties mentioned above, upper continuous images need not. However, upper continuous images are combinatorially easier to handle.

## 8. Productively Lindelöf spaces

We conclude this paper with the following theorems concerning the property of being productively Lindelöf. A topological space has *countable type* if each compact set in $X$ is contained in one of countable outer character. We will use the following lemmata.

**Lemma 8.1** (Alas–Aurichi–Junqueira–Tall). *(See [1].) Let $X$ be a Lindelöf space of countable type. If there is an uncountable set $A \subseteq X$ such that $A \cap K$ is countable for every compact $K \subseteq X$, then $X$ is not productively Lindelöf.*

Improving upon earlier results by several authors, Aurichi and Tall [2] proved that, if $\mathfrak{d} = \aleph_1$, then all productively Lindelöf countable type spaces satisfy $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$. The following theorem both strengthens and generalizes this result.

**Theorem 8.2** ($\mathfrak{d} = \aleph_1$). *Every productively Lindelöf metric* (*or just countable type*) *space is*:

(1) *Productively $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$*;
(2) *Productively $\mathsf{S}_{\text{fin}}(\Omega, \Omega^{\text{gp}})$;*
(3) *Productively $\mathsf{S}_{\text{fin}}(\mathrm{O}, \mathrm{O})$*;
(4) *Productively $\mathsf{S}_{\text{fin}}(\Omega, \Omega)$*; *and*
(5) *Productively $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Omega)$.*

**Proof.** (2) follows from (1), and (4) from (3).

Let $X$ be a productively Lindelöf space and let $\{s_\alpha\colon \alpha < \aleph_1\} \subseteq \mathbb{N}^{\mathbb{N}}$ be a scale.

(1) Assume that $Y$ is $\mathsf{U}_{\text{fin}}(\mathrm{O}, \Gamma)$ and $X \times Y$ is not. By Theorem 7.3, there is an upper continuous $\Psi : X \times Y \to \mathbb{N}^{\mathbb{N}}$ such that $\Psi[X \times Y]$ is unbounded. For each $\alpha < \aleph_1$, pick $(x_\alpha, y_\alpha) \in X \times Y$ such that

$$s_\alpha(n) \leqslant \Psi(x_\alpha, y_\alpha)(n)$$

for infinitely many $n$. Let $A = \{x_\alpha\colon\ \alpha < \aleph_1\}$. Let $K \subseteq X$ be compact. Then $K \times Y$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$, and thus $\Psi[K \times Y]$ is bounded by some $g \in \mathbb{N}^{\mathbb{N}}$. Let $\alpha < \aleph_1$ be such that $g <^* s_\alpha$. For each $\beta \geqslant \alpha$,

$$g(n) < s_\alpha(n) \leqslant s_\beta(n) \leqslant \Psi(x_\beta, y_\beta)(n),$$

for infinitely many $n$, and therefore $(x_\beta, y_\beta) \notin K \times Y$. In particular, $A \cap K$ is countable. By Lemma 8.1, $X$ is not productively Lindelöf.

(3) Similar to (1): Assume that $Y$ is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ and $X \times Y$ is not. By Theorem 7.5, there is an upper continuous $\Psi : X \times Y \to \mathbb{N}^{\mathbb{N}}$ such that $\Psi[X \times Y]$ is dominating. For each $\alpha < \aleph_1$, pick $(x_\alpha, y_\alpha) \in X \times Y$ with

$$s_\alpha \leqslant^* \Psi(x_\alpha, y_\alpha).$$

Let $A = \{x_\alpha\colon\ \alpha < \aleph_1\}$. Let $K \subseteq X$ be compact. As $K \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$, $\Psi[K \times Y]$ is not dominating. Let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness for that. Let $\alpha < \aleph_1$ be such that $g <^* s_\alpha$. For each $\beta \geqslant \alpha$,

$$g <^* s_\alpha \leqslant^* s_\beta \leqslant^* \Psi(x_\beta, y_\beta),$$

and therefore $(x_\beta, y_\beta) \notin K \times Y$. Thus, $A \cap K$ is countable, and by Lemma 8.1, $X$ is not productively Lindelöf.

(5) Assume that $Y$ is $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Omega)$ and $X \times Y$ is not. By Theorem 7.4, there is an upper continuous $\Psi : X \times Y \to \mathbb{N}^{\mathbb{N}}$ such that $\Psi[X \times Y]$ is finitely dominating. Then there is $k$ such that $\Psi[X \times Y]$ is $k$-dominating, that is, for each $f \in \mathbb{N}^{\mathbb{N}}$ there are $(x_1, y_1), \ldots, (x_k, y_k) \in X \times Y$ such that $f \leqslant^* \max\{\Psi(x_1, y_1), \ldots, \Psi(x_k, y_k)\}$. For each $\alpha < \aleph_1$, pick $(x_1^\alpha, y_1^\alpha), \ldots, (x_k^\alpha, y_k^\alpha) \in X \times Y$ with

$$s_\alpha \leqslant^* \max\bigl\{\Psi\bigl(x_1^\alpha, y_1^\alpha\bigr), \ldots, \Psi\bigl(x_k^\alpha, y_k^\alpha\bigr)\bigr\}.$$

Define an upper continuous map

$$\begin{aligned} \Phi : X^k \times Y &\to \mathbb{N}^{\mathbb{N}} \\ (x_1, \ldots, x_k, y) &\mapsto \max\bigl\{\Psi(x_1, y), \ldots, \Psi(x_k, y)\bigr\}. \end{aligned}$$

Let $A = \{(x_1^\alpha, \ldots, x_k^\alpha)\colon\ \alpha < \aleph_1\} \subseteq X^k$. Let $K \subseteq X^k$ be compact. As $K \times Y$ satisfies $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Omega)$, $\Phi[K \times Y]$ is not finitely dominating. Let $g \in \mathbb{N}^{\mathbb{N}}$ be a witness for that. Let $\alpha < \aleph_1$ be such that $g <^* s_\alpha$. For each $\beta \geqslant \alpha$,

$$\begin{aligned} g <^* s_\alpha \leqslant^* s_\beta &\leqslant^* \max\bigl\{\Psi\bigl(x_1^\beta, y_1^\beta\bigr), \ldots, \Psi\bigl(x_k^\beta, y_k^\beta\bigr)\bigr\} \\ &\leqslant^* \max\bigl\{\Phi\bigl(x_1^\beta, \ldots, x_k^\beta, y_1^\beta\bigr), \ldots, \Phi\bigl(x_1^\beta, \ldots, x_k^\beta, y_k^\beta\bigr)\bigr\}, \end{aligned}$$

and therefore $(x_1^\beta, \ldots, x_k^\beta) \notin K$. Thus, $A \cap K$ is countable, and by Lemma 8.1, $X^k$ is not productively Lindelöf. It follows that $X$ is not productively Lindelöf. $\square$

Say that a topological space $X$ is $\mathfrak{b}$-*scalefully* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ if for each upper continuous map $\Psi : X \to \mathbb{N}^{\mathbb{N}}$ and each $\mathfrak{b}$-scale $B = \{b_\alpha\colon\ \alpha < \mathfrak{b}\}$, there is $\alpha < \mathfrak{b}$ such that for each $x \in X$, $\Psi(x)(n) \leqslant b_\alpha(n)$ for infinitely many $n$. Every $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ space is $\mathfrak{b}$-scalefully $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ and every $\mathfrak{b}$-scalefully $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ space is $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

If every $\mathfrak{b}$-scale is dominating (this holds, for example, in the Laver model) then every $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ space is $\mathfrak{b}$-scalefully $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. One can prove, in ZFC, that if every $\mathfrak{b}$-scale is dominating, then $\aleph_1 < \mathfrak{b} = \mathfrak{d}$. On the other hand, if $\mathfrak{b} < \mathfrak{d}$ then every $\mathfrak{b}$-scale is an example of an $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ space that is not $\mathfrak{b}$-scalefully $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

**Theorem 8.3** ($\mathfrak{b} = \aleph_1$). *Every productively Lindelöf metric (or just countable type) space is productively $\mathfrak{b}$-scalefully* $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

**Proof.** Let $Y$ be a $\mathfrak{b}$-scalefully $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$ space. Suppose, contrary to our claim, that $X \times Y$ is not $\mathfrak{b}$-scalefully $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. Then there are an upper continuous map $\Psi : X \times Y \to \mathbb{N}^{\mathbb{N}}$ and a $\mathfrak{b}$-scale $B = \{b_\alpha\colon \alpha < \mathfrak{b}\}$ such that, for each $\alpha < \aleph_1$, there is $(x_\alpha, y_\alpha) \in X \times Y$ such that $b_\alpha \leqslant^* \Psi(x_\alpha, y_\alpha)$.

Let $A = \{x_\alpha\colon \alpha < \aleph_1\}$. By Lemma 8.1 it is enough to prove that $A \cap K$ is countable for all compact $K \subseteq X$. Indeed, it is easy to see that $\Psi$ witnesses that, for every $B$ such that $|B \cap A| = \aleph_1$, $B \times Y$ is not $\mathfrak{b}$-scalefully $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. It suffices to observe that every compact space is productively $\mathfrak{b}$-scalefully $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. $\square$

Alas, Aurichi, Junqueira, and Tall proved in [1] that, if $\mathfrak{b} = \aleph_1$, then every productively Lindelöf countable type space satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$. We obtain a stronger result.

**Corollary 8.4** ($\mathfrak{b} = \aleph_1$). *Let $X$ be a productively Lindelöf metric (or just countable type) space. For each $\mathsf{U}_{\mathrm{fin}}(\mathrm{O}, \Gamma)$ space $Y$, the product space $X \times Y$ satisfies $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.* $\square$

It is known [23] that, if there is a Michael space, then all productively Lindelöf spaces satisfy $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$.

**Problem 8.5.** Assume that there is a Michael space. Is every productively Lindelöf space productively $\mathsf{S}_{\mathrm{fin}}(\mathrm{O}, \mathrm{O})$?

## Acknowledgements

Parts of the work reported here were carried out during two visits of the second named author at the Kurt Gödel Research Center. These visits were partially supported by FWF Grant M1244-N13 and by the Research Networking Programme *New Frontiers of Infinity* (INFTY), funded by the European Science Foundation. The second named author thanks the third named author for his kind hospitality, and the Kurt Gödel Research Center Director, researchers and staff for the excellent academic and friendly atmosphere. We thank the referee for proposing a stronger version of Lemma 2.1.

## References

[1] O. Alas, L. Aurichi, L. Junqueira, F. Tall, Non-productively Lindelöf spaces and small cardinals, Houst. J. Math. 37 (2011) 1373–1381.
[2] L. Aurichi, F. Tall, Lindelöf spaces which are indestructible, productive, or D, Topol. Appl. 159 (2012) 331–340.
[3] L. Babinkostova, M. Scheepers, Products and selection principles, Topol. Proc. 31 (2007) 1–13.
[4] T. Bartoszyński, H. Judah, On cofinality of the smallest covering of the real line by meager sets II, Proc. Am. Math. Soc. 123 (1995) 1879–1885.
[5] T. Bartoszyński, H. Judah, Set Theory: On the Structure of the Real Line, A.K. Peters, Massachusetts, 1995.
[6] T. Bartoszyński, S. Shelah, Continuous images of sets of reals, Topol. Appl. 116 (2001) 243–253.
[7] T. Bartoszyński, B. Tsaban, Hereditary topological diagonalizations and the Menger–Hurewicz Conjectures, Proc. Am. Math. Soc. 134 (2006) 605–615.
[8] T. Bartoszyński, S. Shelah, B. Tsaban, Additivity properties of topological diagonalizations, J. Symb. Log. 68 (2003) 1254–1260.
[9] A. Besicovitch, Concentrated and rarified sets of points, Acta Math. 62 (1934) 289–300.
[10] A. Besicovitch, Relations between concentrated sets and sets possessing property C, Math. Proc. Camb. Philos. Soc. 38 (1942) 20–23.
[11] A. Blass, Near coherence of filters. I. Cofinal equivalence of models of arithmetic, Notre Dame J. Form. Log. 27 (1986) 579–591.
[12] A. Blass, Combinatorial cardinal characteristics of the continuum, in: M. Foreman, A. Kanamori (Eds.), Handbook of Set Theory, Springer, 2010, pp. 395–489.
[13] A. Blass, H. Mildenberger, On the cofinality of ultrapowers, J. Symb. Log. 64 (1999) 727–736.
[14] W. Just, A. Miller, M. Scheepers, P. Szeptycki, The combinatorics of open covers II, Topol. Appl. 73 (1996) 241–266.
[15] A. Kechris, Classical Descriptive Set Theory, Springer, 1995.

[16] L. Kočinac, Selected results on selection principles, in: Sh. Rezapour (Ed.), Proceedings of the 3rd Seminar on Geometry and Topology, Tabriz, Iran, July 15–17, 2004, pp. 71–104.
[17] L. Kočinac, M. Scheepers, Combinatorics of open covers (VII): Groupability, Fundam. Math. 179 (2003) 131–155.
[18] H. Mildenberger, Saharon Shelah, The principle of near coherence of filters does not imply the filter dichotomy principle, Trans. Am. Math. Soc. 361 (2009) 2305–2317.
[19] H. Mildenberger, S. Shelah, B. Tsaban, Covering the Baire space by families which are not finitely dominating, Ann. Pure Appl. Log. 140 (2006) 60–71.
[20] A. Miller, B. Tsaban, Point-cofinite covers in Laver's model, Proc. Am. Math. Soc. 138 (2010) 3313–3321.
[21] T. Orenshtein, B. Tsaban, Linear $\sigma$-additivity and some applications, Trans. Am. Math. Soc. 363 (2011) 3621–3637.
[22] I. Recław, Every Luzin set is undetermined in the point-open game, Fundam. Math. 144 (1994) 43–54.
[23] D. Repovš, L. Zdomskyy, On the Menger covering property and D-spaces, Proc. Am. Math. Soc. 140 (2012) 1069–1074.
[24] D. Repovš, B. Tsaban, L. Zdomskyy, Hurewicz sets of reals without perfect subsets, Proc. Am. Math. Soc. 136 (2008) 2515–2520.
[25] M. Sakai, Property $C''$ and function spaces, Proc. Am. Math. Soc. 104 (1988) 917–919.
[26] N. Samet, M. Scheepers, B. Tsaban, Partition relations for Hurewicz-type selection hypotheses, Topol. Appl. 156 (2009) 616–623.
[27] M. Scheepers, Combinatorics of open covers I: Ramsey theory, Topol. Appl. 69 (1996) 31–62.
[28] M. Scheepers, Selection principles and covering properties in topology, Note Mat. 22 (2003) 3–41.
[29] M. Scheepers, B. Tsaban, The combinatorics of Borel covers, Topol. Appl. 121 (2002) 357–382.
[30] B. Tsaban, A diagonalization property between Hurewicz and Menger, Real Anal. Exch. 27 (2001/2002) 757–763.
[31] B. Tsaban, Some new directions in infinite-combinatorial topology, in: J. Bagaria, S. Todorčevic (Eds.), Set Theory, in: Trends in Mathematics, Birkhäuser, 2006, pp. 225–255.
[32] B. Tsaban, $o$-bounded groups and other topological groups with strong combinatorial properties, Proc. Am. Math. Soc. 134 (2006) 881–891.
[33] B. Tsaban, Additivity numbers of covering properties, in: L. Kočinac (Ed.), Selection Principles and Covering Properties in Topology, in: Quaderni di Matematica, vol. 18, Seconda Universita di Napoli, Caserta, 2006, pp. 245–282.
[34] B. Tsaban, Menger's and Hurewicz's Problems: Solutions from "The Book" and Refinements, Contemporary Mathematics, vol. 533, 2011, pp. 211–226.
[35] B. Tsaban, L. Zdomskyy, Scales, fields, and a problem of Hurewicz, J. Eur. Math. Soc. 10 (2008) 837–866.
[36] B. Tsaban, L. Zdomskyy, Combinatorial images of sets of reals and semifilter trichotomy, J. Symb. Log. 73 (2008) 1278–1288.
[37] B. Tsaban, L. Zdomskyy, Additivity of the Gerlits–Nagy property and concentrated sets, Proc. Am. Math. Soc. (2014), in press.
[38] L. Zdomskyy, A semifilter approach to selection principles, Comment. Math. Univ. Carol. 46 (2005) 525–539.