Products of Menger spaces: A combinatorial approach

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We construct Menger subsets of the real line whose product is not Menger in the plane. In contrast to earlier constructions, our approach is purely combinatorial. The set theoretic hypothesis used in our construction is far milder than earlier ones, and holds in almost all canonical models of set theory of the real line. On the other hand, we establish productive properties for versions of Menger's property parameterized by filters and semifilters. In particular, the Continuum Hypothesis implies that every productively Menger set of real numbers is productively Hurewicz, and each ultrafilter version of Menger's property is strictly between Menger's and Hurewicz's classic properties. We include a number of open problems emerging from this study.

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1. Introduction

A topological space X is *Menger* if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \dots$ of open covers of the space X, there are finite subsets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2, \dots$ whose union forms a cover of the space X. This property was introduced by Karl Menger [17], and reformulated as presented here by Witold Hurewicz [11]. Menger's property is strictly between σ-compact and Lindelöf. Now a central notion in topology, it has applications in a number of branches of topology and set theory. The undefined notions in the following example, which are available in the indicated references, are not needed for the remainder of this paper.

Example 1.1. Menger spaces form the most general class for which a positive solution of the D-space problem is known [2, Corollary 2.7], and the most general class for which a general form of Hindman's Finite Sums

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Theorem holds [27]. In set theory, Menger's property characterizes filters whose Mathias forcing notion does not add dominating functions [9].

Menger's property is hereditary for closed subsets and continuous images. By a classic result of Todorčević there are, provably, Menger spaces X and Y such that the product space $X \times Y$ is not Menger [24, §3]. It remains open whether there are, provably, such examples in the real line, or even just metrizable examples [25, Problem 6.7]. This problem, proposed by Scheepers long ago, resisted tremendous efforts thus far.

For brevity, sets of real numbers are called here *real sets*.[1] An uncountable real set is *Luzin* if its intersection with every meager (Baire first category) set is countable. Assuming the Continuum Hypothesis, there are two Luzin sets whose product is not Menger [12, Theorem 3.7]. An uncountable real set X is *concentrated* if it has a countable subset D such that the set $X \setminus U$ is countable for every open set U containing D. Every Luzin set is concentrated, and every concentrated set has Menger's property. This approach extends to obtain similar examples using a set theoretic hypothesis about the meager sets that is weaker than the Continuum Hypothesis [21, Theorem 49]. Later methods [29, Theorem 9.1] were combined with reasoning on meager sets to obtain examples using another portion of the Continuum Hypothesis [19, Theorem 3.3].

We introduce a purely combinatorial approach to products of Menger sets. We obtain examples using hypotheses milder than earlier ones, as well as examples using hypotheses that are incompatible with the Continuum Hypothesis. To this end, we introduce the key notion of *bi-∂-unbounded set*, and determine the limits on its possible existence. We extend these results to variations of Menger's property parameterized by filters and semifilters (defined below). For a semifilter S, we introduce the notion of S-*scale*. These scales provably exist, and capture a number of distinct special cases used in earlier works.

The second part of the paper, beginning with Section 5, establishes provably productive properties among semifilter-parameterized Menger properties. If S is an *ultrafilter*, then every S-scale gives rise to a real set that is productively S-Menger. We deduce that each of these variations of Menger's property is strictly between Hurewicz's and Menger's classic properties.

The last section includes a discussion of related results and open problems suggested by this study.

2. Products of Menger sets

Towards a combinatorial treatment of the questions discussed here, we identify the Cantor space $\{0,1\}^{\mathbb{N}}$ with the family $P(\mathbb{N})$ of all subsets of the set $\mathbb{N}$. Since the Cantor space is homeomorphic to Cantor's set, every subspace of the space $P(\mathbb{N})$ is considered as a real set.

The space $P(\mathbb{N})$ splits into two important subspaces: the family of infinite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^{\infty}$, and the family of finite subsets of $\mathbb{N}$, denoted $[\mathbb{N}]^{<\infty}$. We identify every set $a \in [\mathbb{N}]^{\infty}$ with its increasing enumeration, an element of the Baire space $\mathbb{N}^{\mathbb{N}}$. Thus, for a natural number n, $a(n)$ is the n-th element in the increasing enumeration of the set a. This way, we have $[\mathbb{N}]^{\infty} \subseteq \mathbb{N}^{\mathbb{N}}$, and the topology of the space $[\mathbb{N}]^{\infty}$ (a subspace of the Cantor space $P(\mathbb{N})$) coincides with the subspace topology induced by $\mathbb{N}^{\mathbb{N}}$. This explains some of the elementary assertions made here; moreover, notions defined here for $[\mathbb{N}]^{\infty}$ are often adaptations of classic notions for $\mathbb{N}^{\mathbb{N}}$. Depending on the interpretation, points of the space $[\mathbb{N}]^{\infty}$ are referred to as sets or functions.

For functions $a, b \in [\mathbb{N}]^{\infty}$, we write $a \leq b$ if $a(n) \leq b(n)$ for all natural numbers n, and $a \leq^{*} b$ if $a(n) \leq b(n)$ for almost all natural numbers n, that is, the set of exceptions $\{\, n : b(n) < a(n) \,\}$ is finite. We follow the convention that *bounded* means *has an upper bound* in the ambient superset.

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[1] The term *real set* is a natural extension of the standard notions *real number*, *real matrix*, *real function*, etc., and should be understood as a convenient abbreviation. It does not imply that other sets are less "real".

Definition 2.1. Let κ be an infinite cardinal number. A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq \kappa$ is *κ-unbounded* if the cardinality of every $\leq$-bounded subset of the set X is smaller than κ.

Remark 2.2. For cardinal numbers κ of uncountable cofinality, which will be the case in the present paper, the notion of κ-unbounded defined here is equivalent to its variation using the relation $\leq^*$ instead of $\leq$. This is not the case for cardinal numbers of countable cofinality.

Let κ be an infinite cardinal number. A topological space X with $|X| \geq \kappa$ is *κ-concentrated* on a countable set $D \subseteq X$ if $|X \setminus U| < \kappa$ for all open sets U containing D.

Every compact set $K \subseteq [\mathbb{N}]^\infty$ is $\leq$-bounded. A classic argument of Lawrence [15, Propositions 2–3] implies that, for each cardinal number κ, the existence of a κ-concentrated real set is equivalent to the existence of a κ-unbounded set in $[\mathbb{N}]^\infty$. Essentially, this is due to the following fact.

Lemma 2.3. *Let κ be a cardinal number, and $X \subseteq [\mathbb{N}]^\infty$ be a set with $|X| \geq \kappa$. The set X is κ-unbounded if and only if the real set $X \cup [\mathbb{N}]^{<\infty}$ is κ-concentrated on $[\mathbb{N}]^{<\infty}$.*

Proof. ($\Rightarrow$) Let $U \subseteq P(\mathbb{N})$ be an open set containing the set $[\mathbb{N}]^{<\infty}$. The set $K := P(\mathbb{N}) \setminus U$ is a closed, and thus compact, subset of $P(\mathbb{N})$. Since $U \supseteq [\mathbb{N}]^{<\infty}$, we have $K \subseteq [\mathbb{N}]^\infty$. Since compact subsets of $[\mathbb{N}]^\infty$ are $\leq$-bounded and the set X is κ-unbounded, we have

$$\left| (X \cup [\mathbb{N}]^{<\infty}) \cap K \right| = |X \cap K| < \kappa.$$

($\Leftarrow$) For each bound $b \in [\mathbb{N}]^\infty$, the set $K := \{\, a \in [\mathbb{N}]^\infty : a \leq b \,\}$ is compact. Thus, the set $U := P(\mathbb{N}) \setminus K$ is an open set containing $[\mathbb{N}]^{<\infty}$, and we have $|X \setminus U| < \kappa$. $\square$

A set $X \subseteq [\mathbb{N}]^\infty$ is *dominating* if for each function $a \in [\mathbb{N}]^\infty$ there is a function $x \in X$ such that $a \leq^* x$. Let $\mathfrak{d}$ be the minimal cardinality of a dominating set in $[\mathbb{N}]^\infty$. Much information about the cardinal number $\mathfrak{d}$, and about other ones defined below, is available [7]. Every real set of cardinality smaller than $\mathfrak{d}$ is Menger, and no dominating subset of $[\mathbb{N}]^\infty$ is Menger [12, Theorem 4.4]. The former assertion implies that every $\mathfrak{d}$-concentrated real set is Menger.[2]

Corollary 2.4. *For each $\mathfrak{d}$-unbounded set $X \subseteq [\mathbb{N}]^\infty$, the real set $X \cup [\mathbb{N}]^{<\infty}$ is Menger.* $\square$

Definition 2.5. For functions $a, b \in [\mathbb{N}]^\infty$, we write $a \leq^\infty b$ if $b \not\leq^* a$, that is, if $a(n) \leq b(n)$ for infinitely many natural numbers n. For a set $X \subseteq [\mathbb{N}]^\infty$ and a function $b \in [\mathbb{N}]^\infty$, we write $X \leq^\infty b$ if $x \leq^\infty b$ for each function $x \in X$. This convention applies to all binary relations.

There are, provably, $\mathfrak{d}$-unbounded sets and $\mathrm{cf}(\mathfrak{d})$-unbounded sets: Let $\{\, d_\alpha : \alpha < \mathfrak{d} \,\}$ be a dominating set. For each ordinal number $\alpha < \mathfrak{d}$, take a function $x_\alpha \in [\mathbb{N}]^\infty$ such that $\{\, d_\beta, x_\beta : \beta < \alpha \,\} <^\infty x_\alpha$. Then the set $\{\, x_\alpha : \alpha < \mathfrak{d} \,\}$ is $\mathfrak{d}$-unbounded. Taking a cofinal subset $I \subseteq \mathfrak{d}$ of cardinality $\mathrm{cf}(\mathfrak{d})$, we obtain the $\mathrm{cf}(\mathfrak{d})$-unbounded set $\{\, x_\alpha : \alpha \in I \,\}$.

Lemma 2.6. *For sets $a, b \in P(\mathbb{N})$, let*

$$a \uplus b := (2a) \cup (2b+1) = \{\, 2k : k \in a \,\} \cup \{\, 2k+1 : k \in b \,\}.$$

Then:

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[2] Moreover, $\mathfrak{d}$-concentrated sets have the stronger selective property $\mathsf{S}_1(\Gamma, \mathsf{O})$ [6,26].

(1) *For each set $a \in [\mathbb{N}]^{\infty}$ and each natural number n, we have $(a \uplus a)(2n) = 2a(n) + 1$.*
(2) *For all sets $a, b, c, d \in [\mathbb{N}]^{\infty}$ with $a \leq b$ and $c \leq d$, we have $a \uplus c \leq b \uplus d$.* $\square$

Theorem 2.7. *Let $\kappa \in \{\mathrm{cf}(\mathfrak{d}), \mathfrak{d}\}$, and $X \subseteq [\mathbb{N}]^{\infty}$ be a set containing a κ-unbounded set. There is a $\mathfrak{d}$-concentrated real set Y such that the planar set $X \times Y$ is not Menger.*

Proof. Let $A \subseteq X$ be a κ-unbounded set. By moving to a subset of A, we may assume that $|A| = \kappa$. Let $D \subseteq [\mathbb{N}]^{\infty}$ be a dominating set of cardinality $\mathfrak{d}$. Decompose

$$D = \bigcup_{a \in A} I_a$$

such that $\left| \bigcup_{a \in B} I_a \right| < \mathfrak{d}$ for all sets $B \subseteq A$ of cardinality smaller than κ. (If $\kappa = \mathfrak{d}$, we can take every set I_a to be a singleton.) Fix elements $a \in A$ and $d \in I_a$. Take a function $d' \in [\mathbb{N}]^{\infty}$ such that $a, d \leq d'$. Consider the set $\{a \uplus d' : a \in A, d \in I_a\}$. Its cardinality is at most $\mathfrak{d}$, and since its projection on the odd coordinates is dominating, its cardinality is exactly $\mathfrak{d}$.

Claim 2.8. *The set $\{a \uplus d' : a \in A, d \in I_a\}$ is $\mathfrak{d}$-unbounded.*

Proof. Let $b \in [\mathbb{N}]^{\infty}$. Define $b'(n) := b(2n)$ for all natural numbers n. Let $K := \{a \in A : a \leq b'\}$. Then $|K| < \kappa$.

Let $a \in A \setminus K$ and $d \in I_a$. There is a natural number n such that

$$b(2n) = b'(n) < a(n) \leq 2a(n) + 1 = (a \uplus a)(2n) \leq (a \uplus d')(2n),$$

and thus $a \uplus d' \not\leq b$. Therefore,

$$|\{a \uplus d' : a \in A, d \in I_a, a \uplus d' \leq b\}| \leq |\{a \uplus d' : a \in K, d \in I_a\}| < \mathfrak{d}. \quad \square$$

By [Lemma 2.3](), the real set

$$Y := \{a \uplus d' : a \in A, d \in I_a\} \cup [\mathbb{N}]^{<\infty}$$

is $\mathfrak{d}$-concentrated on the set $[\mathbb{N}]^{<\infty}$. In particular, the set Y is Menger.

For sets $a, b \in \mathrm{P}(\mathbb{N})$, let $a \oplus b$ denote the symmetric difference of the sets a and b. With respect to the operator $\oplus$, the space $\mathrm{P}(\mathbb{N})$ is a topological group.

Claim 2.9. *The set $(2X) \oplus Y$ is a dominating subset of $[\mathbb{N}]^{\infty}$.*

Proof. For all sets $a, b, c \in \mathrm{P}(\mathbb{N})$, we have $(2a) \oplus (b \uplus c) = (a \oplus b) \uplus c \supseteq 2c + 1$. It follows that $(2X) \oplus Y \subseteq [\mathbb{N}]^{\infty}$.

For each function d in the dominating set D we started with, let $a \in A$ be a function such that $d \in I_a$. As $a \in X$ and $a \uplus d' \in Y$, we have

$$2a \oplus (a \uplus d') = (a \oplus a) \uplus d' = \emptyset \uplus d' = 2d' + 1 \in (2X) \oplus Y.$$

Since $d \leq d' \leq 2d' + 1$ for all functions $d \in D$, the set $(2X) \oplus Y$ is dominating. $\square$

In summary, the set $(2X) \oplus Y$ is a continuous image of the planar set $X \times Y$ in $[\mathbb{N}]^{\infty}$ that is dominating. It follows that the space $X \times Y$ is not Menger. $\square$

Let X be a real set of cardinality smaller than $\mathfrak{d}$. Then the set X is *trivially* Menger: the topology used is irrelevant, as long as we restrict attention to countable covers. In particular, all finite powers of the set X are Menger, even for countable *Borel* covers; a strong property [21].

Theorem 2.10. *Assume that* $\mathrm{cf}(\mathfrak{d}) < \mathfrak{d}$. *There are real sets* X *and* Y *such that* $|X| < \mathfrak{d}$ *and the set* Y *is* $\mathfrak{d}$-*concentrated, but the planar set* $X \times Y$ *is not Menger.*

Proof. By the discussion preceding Lemma 2.6, there are $\mathrm{cf}(\mathfrak{d})$-unbounded sets in $[\mathbb{N}]^\infty$. Apply Theorem 2.7 to any of these sets. □

Let κ be a cardinal number. A real set of cardinality at least κ is κ-*Luzin* if the cardinalities of its intersections with meager sets are all smaller than κ. Let $\mathrm{cov}(\mathcal{M})$ be the minimal cardinality of a cover of the real line by meager sets, and $\mathrm{cof}(\mathcal{M})$ be the minimal cardinality of a cofinal family of meager real sets. The hypothesis $\mathrm{cov}(\mathcal{M}) = \mathrm{cof}(\mathcal{M})$ implies that there are $\mathrm{cov}(\mathcal{M})$-Luzin sets whose product is not Menger [21, Theorem 49]. Since $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d}$, every $\mathrm{cov}(\mathcal{M})$-Luzin set is $\mathfrak{d}$-concentrated, and thus Menger. In general, $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{d} \leq \mathrm{cof}(\mathcal{M})$, and thus the following corollary implies (using the same hypothesis) that for *every* $\mathrm{cov}(\mathcal{M})$-Luzin set L there is a $\mathfrak{d}$-concentrated real set Y such that the planar set $L \times Y$ is not Menger.

Corollary 2.11. *Let* $\kappa \in \{\mathrm{cf}(\mathfrak{d}), \mathfrak{d}\}$. *For each* κ-*Luzin set* L, *there is a* $\mathfrak{d}$-*concentrated real set* Y *such that the planar set* $L \times Y$ *is not Menger. In particular, if* $\aleph_1 = \mathrm{cf}(\mathfrak{d})$, *then this is the case for every Luzin set.*

Proof. By applying a homeomorphism, we may assume that $L \subseteq [\mathbb{N}]^\infty$. Every κ-Luzin subset of $[\mathbb{N}]^\infty$ is κ-unbounded. Apply Theorem 2.7. □

The most important application of Theorem 2.7 appears in the next section.

3. Bi-$\mathfrak{d}$-unbounded sets

For a set $a \in \mathrm{P}(\mathbb{N})$, let $a^{\mathsf{c}} := \mathbb{N} \setminus a$. Let $[\mathbb{N}]^{\infty,\infty} := \{\, a \in [\mathbb{N}]^\infty : a^{\mathsf{c}} \in [\mathbb{N}]^\infty \,\}$, the family of infinite co-infinite subsets of $\mathbb{N}$.

Definition 3.1. Let κ be an infinite cardinal number. A set $X \subseteq [\mathbb{N}]^{\infty,\infty}$ is *bi-κ-unbounded* if the sets X and $\{\, x^{\mathsf{c}} : x \in X \,\} \subseteq [\mathbb{N}]^\infty$ are both κ-unbounded.

Theorem 3.2. *Let* $\kappa \in \{\mathrm{cf}(\mathfrak{d}), \mathfrak{d}\}$. *Let* $X \subseteq [\mathbb{N}]^\infty$ *be a bi-κ-unbounded set. Then:*

(1) *The real set* $X \cup [\mathbb{N}]^{<\infty}$ *is* κ-*concentrated. In particular, it is Menger.*
(2) *There is a* $\mathfrak{d}$-*concentrated real set* Y *such that the planar set* $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ *is not Menger.*

Proof. (1) By Corollary 2.4.
(2) The continuous image $\{\, x^{\mathsf{c}} : x \in X \cup [\mathbb{N}]^{<\infty} \,\}$ of the set $X \cup [\mathbb{N}]^{<\infty}$ in $\mathrm{P}(\mathbb{N})$ is a κ-unbounded subset of $[\mathbb{N}]^\infty$. Apply Theorem 2.7. □

The existence of bi-$\mathfrak{d}$-unbounded sets and bi-$\mathrm{cf}(\mathfrak{d})$-unbounded sets is a mild hypothesis. A set $r \in [\mathbb{N}]^\infty$ *reaps* a family $A \subseteq [\mathbb{N}]^\infty$ if, for each set $a \in A$, both sets $a \cap r$ and $a \setminus r$ are infinite. Let $\mathfrak{r}$ be the minimal cardinality of a family $A \subseteq [\mathbb{N}]^\infty$ that no set r reaps.

For natural numbers $n < m$, let $[n, m) := \{n, n+1, \ldots, m-1\}$.

Theorem 3.3. *The following assertions are equivalent:*

(1) $\mathfrak{d} \leq \mathfrak{r}$.
(2) *There are bi-$\mathfrak{d}$-unbounded sets in $[\mathbb{N}]^\infty$.*
(3) *There are bi-$\mathrm{cf}(\mathfrak{d})$-unbounded sets in $[\mathbb{N}]^\infty$.*

Proof. $(1) \Rightarrow (2), (3)$: We use the following lemma, to which we provide a short, direct proof.

Lemma 3.4 *(Mejía [16]). Let $X \subseteq [\mathbb{N}]^\infty$. If $|X| < \min\{\mathfrak{d}, \mathfrak{r}\}$, then there is an element $b \in [\mathbb{N}]^{\infty,\infty}$ such that $X \leq^\infty b$ and $X \leq^\infty b^\mathrm{c}$.*

Proof. For a set $x \in [\mathbb{N}]^\infty$ with $1 \notin x$, define a function $\tilde{x} \in [\mathbb{N}]^\infty$ by $\tilde{x}(1) := x(1)$, and $\tilde{x}(n+1) := x(\tilde{x}(n))$ for each natural number n.

We may assume that $1 \notin x$ for all sets $x \in X$. Since $|X| < \mathfrak{d}$, there is a function $a \in [\mathbb{N}]^\infty$ such that the sets

$$I_x := \{\, n : |[a(n), a(n+1)) \cap \tilde{x}| \geq 2 \,\}$$

are infinite for all sets $x \in X$ [7, Theorem 2.10]. Since $|X| < \mathfrak{r}$, there is a set $r \in [\mathbb{N}]^\infty$ that reaps the family $\{\, I_x : x \in X \,\}$. Define

$$b := \bigcup_{n \in r} [a(n), a(n+1)).$$

Fix a set $x \in X$. Let n be a member of the infinite set $r \cap I_x$. There are at least two elements in the set $[a(n), a(n+1)) \cap \tilde{x}$; let $\tilde{x}(i)$ be the minimal one. Then $x(\tilde{x}(i)) = \tilde{x}(i+1) \in [a(n), a(n+1))$. Since $n \in r$, the set $b^\mathrm{c} \cap [a(n), a(n+1))$ is empty, and thus $a(n+1) \leq b^\mathrm{c}(\tilde{x}(i))$. It follows that $x(\tilde{x}(i)) < b^\mathrm{c}(\tilde{x}(i))$. Similarly, every number $n \in I_x \setminus r$ produces a number i such that $x(\tilde{x}(i)) < b(\tilde{x}(i))$. $\quad\square$

Let $\{\, d_\alpha : \alpha < \mathfrak{d} \,\} \subseteq [\mathbb{N}]^\infty$ be a dominating set. By Lemma 3.4, for each ordinal number $\alpha < \mathfrak{d}$, there is a set $x_\alpha \in [\mathbb{N}]^{\infty,\infty}$ such that $\{\, d_\beta, x_\beta : \beta < \alpha \,\} <^\infty x_\alpha, x_\alpha^\mathrm{c}$. Then the set $\{\, x_\alpha : \alpha < \mathfrak{d} \,\}$ is bi-$\mathfrak{d}$-unbounded. Let I be a cofinal subset of the cardinal number $\mathfrak{d}$, of cardinality $\mathrm{cf}(\mathfrak{d})$. Then the set $\{\, x_\alpha : \alpha \in I \,\}$ is bi-$\mathrm{cf}(\mathfrak{d})$-unbounded.

$(2) \Rightarrow (1)$: We may assume that the cardinal number $\mathfrak{d}$ is regular. Indeed, it is known that if $\mathfrak{r} < \mathfrak{d}$ then $\mathfrak{d}$ is regular.[3] Thus, if $\mathfrak{d}$ is singular, then $\mathfrak{d} \leq \mathfrak{r}$, and we are done.

Let $X \subseteq [\mathbb{N}]^\infty$ be a bi-$\mathfrak{d}$-unbounded set. Let $A \subseteq [\mathbb{N}]^\infty$ be a family with $|A| < \mathfrak{d}$. We prove that the family A is reapable. We may assume that for each set $a \in A$ and each finite set s, we have $a \setminus s \in A$.

Since the set X is bi-$\mathfrak{d}$-unbounded, the set

$$\bigcup_{a \in A} \{\, x \in X : x \leq^* a \text{ or } x \leq^* a^\mathrm{c} \,\}$$

is a union of less than $\mathfrak{d}$ sets, each of cardinality smaller than $\mathfrak{d}$. Thus, there is an element $r \in X$ that is not included in that set, that is, such that $A <^\infty r, r^\mathrm{c}$.

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[3] In the notation of Section 4, fix an ultrafilter U with pseudobase P of cardinality $\mathfrak{r}$ [7, Theorem 9.9], and take a $\leq_U$-dominating set D of cardinality $\mathfrak{b}(U)$, a regular cardinal number. Then the set $\{\, f \circ p : f \in D, p \in P \,\}$ is dominating, and thus $\mathfrak{d} \leq \mathfrak{b}(U)(\leq \mathfrak{d})$.

The set r reaps the family A: Fix a set $a \in A$. Assume that the set $a \cap r$ is finite. Then the set $a' := a \setminus r$ is in A, and thus $a' <^\infty r^c$. But $a' \subseteq r^c$, and thus $r^c \leq a'$; a contradiction. For the same reason, the set $a \setminus r$ is infinite, too.

(3) $\Rightarrow$ (1): If the cardinal number $\mathfrak{d}$ is regular, then the previously established implication applies. And if it is singular, then as explained in the previous implication, we have $\mathfrak{d} \leq \mathfrak{r}$. In either case, we are done. $\square$

A topological space X is *Rothberger* if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X, there are elements $U_1 \in \mathcal{U}_1$, $U_2 \in \mathcal{U}_2$, ... with $X \subseteq \bigcup_n U_n$. Every real set of cardinality smaller than $\mathrm{cov}(\mathcal{M})$ is Rothberger [12, Theorem 4.2], and therefore so is every $\mathrm{cov}(\mathcal{M})$-concentrated real set. Since $\mathrm{cov}(\mathcal{M}) \leq \mathfrak{r}$ [7, Theorem 5.19], we obtain the following result.

Corollary 3.5. *Assume that* $\mathrm{cov}(\mathcal{M}) = \mathfrak{d}$. *Then there are two Rothberger real sets whose product is not Menger.* $\square$

4. Filter-Menger spaces

For sets $a, b \in [\mathbb{N}]^\infty$, we write $a \subseteq^* b$ if the set $a \setminus b$ is finite. A *semifilter* [4] is a set $S \subseteq [\mathbb{N}]^\infty$ such that, for each set $s \in S$ and each set $b \in [\mathbb{N}]^\infty$ with $s \subseteq^* b$, we have $b \in S$.[4] Important examples of semifilters include the maximal semifilter $[\mathbb{N}]^\infty$, the minimal semifilter cF of all cofinite sets, and every nonprincipal ultrafilter on $\mathbb{N}$.

Let S be a semifilter. For functions $a, b \in [\mathbb{N}]^\infty$, let

$$[a \leq b] := \{\, n : a(n) \leq b(n) \,\}.$$

We write $a \leq_S b$ if $[a \leq b] \in S$. Let $\mathfrak{b}(S)$ be the minimal cardinality of a $\leq_S$-unbounded subset of $[\mathbb{N}]^\infty$. For a semifilter S, let $S^+ := \{\, a \in [\mathbb{N}]^\infty : a^c \notin S \,\}$. For all sets $a \in S$ and $b \in S^+$, the intersection $a \cap b$ is infinite. For functions $a, b \in [\mathbb{N}]^\infty$, we have that $a \nleq_S b$ if and only if $b <_{S^+} a$. The κ-unbounded sets presented in the previous sections are instances of the following notion, which generalizes the earlier notion of $\mathfrak{b}(S)$-scale [29, Definition 2.8].

Definition 4.1. Let S be a semifilter. A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq \mathfrak{b}(S)$ is an *S-scale* if, for each function $b \in [\mathbb{N}]^\infty$, there is a function $c \in [\mathbb{N}]^\infty$ such that

$$b \leq_{S^+} c \leq_S x$$

for all but less than $\mathfrak{b}(S)$ functions $x \in X$.

Proposition 4.2 *([29, Lemma 2.9])*. *For each semifilter S there is an S-scale.*

Proof. Let $\{\, b_\alpha : \alpha < \mathfrak{b}(S) \,\} \subseteq [\mathbb{N}]^\infty$ be a $\leq_S$-unbounded set. For each ordinal number $\alpha < \mathfrak{b}(S)$, there is a function $x_\alpha \in [\mathbb{N}]^\infty$ such that $\{\, b_\beta, x_\beta : \beta < \alpha \,\} <_S x_\alpha$. The set $\{\, x_\alpha : \alpha < \mathfrak{b}(S) \,\}$ is an S-scale. Indeed, fix a function $b \in [\mathbb{N}]^\infty$. There is an ordinal number $\beta < \mathfrak{b}(S)$ such that $b_\beta \nleq_S b$, and thus $b \leq_{S^+} b_\beta$. For each ordinal number $\alpha > \beta$, we have $b_\beta \leq_S x_\alpha$. $\square$

Let S be a semifilter, and $b, c, x \in [\mathbb{N}]^\infty$ functions satisfying $b \leq_{S^+} c \leq_S x$. Then the set $[b \leq x]$ contains the intersection $[b \leq c] \cap [c \leq x]$ of an element of S^+ and an element of S. In particular, we have $b \leq^\infty x$.

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[4] Semifilters are normally denoted by calligraphic letters. Here, we view them as sets of points in, and thus subspaces of, the Cantor space $\mathrm{P}(\mathbb{N})$. Thus, we use the standard typefaces, as we do for arbitrary points and sets in topological spaces.

Proposition 4.3. *Let S be a semifilter. Every S-scale is a $\mathfrak{b}(S)$-unbounded subset of $[\mathbb{N}]^\infty$, and thus its union with the set $[\mathbb{N}]^{<\infty}$ is $\mathfrak{b}(S)$-concentrated. In particular, no union of an S-scale and $[\mathbb{N}]^{<\infty}$ is σ-compact [26, Lemma 1.6].* $\square$

Let S be a semifilter. A topological space X is *S-Menger* if for each sequence $\mathcal{U}_1, \mathcal{U}_2, \ldots$ of open covers of X, there are finite sets $\mathcal{F}_1 \subseteq \mathcal{U}_1$, $\mathcal{F}_2 \subseteq \mathcal{U}_2$, … such that $\{\, n : x \in \bigcup \mathcal{F}_n \,\} \in S$ for all points $x \in X$. A topological space is Menger if and only if it is $[\mathbb{N}]^\infty$-Menger. For the filter cF of cofinite sets, the property cF-Menger is the classic *Hurewicz property* [11]. Thus, for every semifilter S, we have the following implications.

$$\text{Hurewicz} \Longrightarrow S\text{-Menger} \Longrightarrow \text{Menger}.$$

A function Ψ from a topological space X into $[\mathbb{N}]^\infty$ is *upper continuous* if the sets $\{\, x \in X : \Psi(x)(n) \leq m \,\}$ are open for all natural numbers n and m. In particular, continuous functions are upper continuous. By earlier methods [18, Theorem 7.3], we have the following result.

Proposition 4.4. *Let X be a topological space, and S be a semifilter. The following assertions are equivalent:*

(1) *The space X is S-Menger.*
(2) *The space X is Lindelöf, and every upper continuous image of X in $[\mathbb{N}]^\infty$ is $\leq_S$-bounded.* $\square$

For especially nice classes of spaces, such as Lindelöf zero-dimensional spaces or real sets, *upper continuous* can be replaced by *continuous* in Proposition 4.4. In general, however, this is not the case: The properties considered here are hereditary for closed subsets. Consider the planar set

$$X := ((\mathbb{R} \setminus \mathbb{Q}) \times [0,1]) \cup (\mathbb{R} \times \{1\}) \subseteq \mathbb{R}^2.$$

This set is not Menger, since the non-Menger set $(\mathbb{R} \setminus \mathbb{Q}) \times \{0\}$ (homeomorphic to $[\mathbb{N}]^\infty$) is closed in X. Since the set X is connected, every *continuous* image of X in $[\mathbb{N}]^\infty$ is a singleton.

For a set $a \in S^+$, let

$$S_a := \{\, c \in [\mathbb{N}]^\infty : \exists s \in S, s \cap a \subseteq^* c \,\},$$

the semifilter generated by the sets $\{\, s \cap a : s \in S \,\}$. The following observation generalizes an earlier result [29, Theorem 2.14].

Proposition 4.5. *Let S be a semifilter, and $X \subseteq [\mathbb{N}]^\infty$ be an S-scale. Every upper continuous image of the real set $X \cup [\mathbb{N}]^{<\infty}$ in $[\mathbb{N}]^\infty$ is $\leq_{S_a}$-bounded for some set $a \in S^+$.*

Proof. Let $\Psi \colon X \cup [\mathbb{N}]^{<\infty} \to [\mathbb{N}]^\infty$ be an upper continuous function. We use the forthcoming Lemma 5.1, in the case $Y = \{0\}$. This special case was, implicitly, established by Bartoszyński and Shelah [5, Lemma 2]. This lemma provides a function $b \in [\mathbb{N}]^\infty$ such that $\Psi(x)(n) \leq b(n)$ for all functions $x \in X$ and all natural numbers n with $b(n) \leq x(n)$. Since the set X is an S-scale, there is a function $c \in [\mathbb{N}]^\infty$ such that $b \leq_{S^+} c \leq_S x$ for all but less than $\mathfrak{b}(S)$ functions $x \in X$. For these points x, we have that $\Psi(x)(n) \leq b(n)$ for all natural numbers $n \in [b \leq c] \cap [c \leq x]$. Take $a := [b \leq c]$.

The image of the remaining points of the set $X \cup [\mathbb{N}]^{<\infty}$ is $\leq_S$-bounded by some member $b' \in [\mathbb{N}]^\infty$. Then any function $d \in [\mathbb{N}]^\infty$ with $b, b' \leq d$ is a bound as required. $\square$

By *filter* we mean a semifilter closed under finite intersections. If F is a filter, then $a \cap b \in F^+$ for all sets $a \in F$ and $b \in F^+$. And if F is an ultrafilter, then $F^+ = F$.

Corollary 4.6. *For every filter F, the union of every F-scale and $[\mathbb{N}]^{<\infty}$ is F^+-Menger, and if F is an ultrafilter, this union is F-Menger.* $\quad \square$

Let $\mathfrak{b}$ be the minimal cardinality of a $\leq^*$-unbounded subset of $[\mathbb{N}]^\infty$.

Theorem 4.7. *Assume that $\mathfrak{b} = \mathfrak{d}$. Let S be a semifilter. The following assertions are equivalent:*

(1) *The semifilter S is nonmeager.*
(2) *There are an S-scale $X \subseteq [\mathbb{N}]^\infty$ and a $\mathfrak{d}$-concentrated real set Y such that the planar set $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ is not Menger.*

Proof. $(1) \Rightarrow (2)$: Let $\{ d_\alpha : \alpha < \mathfrak{d} \}$ be a dominating subset of $[\mathbb{N}]^\infty$. Fix an ordinal number $\alpha < \mathfrak{d}$. Since $\mathfrak{b} = \mathfrak{d}$, there is a function $b \in [\mathbb{N}]^\infty$ such that $\{ d_\beta, x_\beta : \beta < \alpha \} <^* b$. The set $\{ x \in [\mathbb{N}]^{\infty,\infty} : b \leq^\infty x^{\mathsf{c}} \}$ is comeager. Since the semifilter S is nonmeager, the set $\{ x \in [\mathbb{N}]^{\infty,\infty} : b \leq_S x \}$ is nonmeager [29, Corollary 3.4]. Thus, there is a set $x_\alpha \in [\mathbb{N}]^{\infty,\infty}$ such that $b \leq_S x_\alpha$ and $b \leq^\infty x_\alpha^{\mathsf{c}}$. Then the set $X := \{ x_\alpha : \alpha < \mathfrak{d} \}$ is an S-scale, and it is bi-$\mathfrak{d}$-unbounded. Apply Theorem 2.7.

$(2) \Rightarrow (1)$: Let S be a meager semifilter, and $X \subseteq [\mathbb{N}]^\infty$ be an S-scale. By Theorem 5.4 below, the set $X \cup [\mathbb{N}]^{<\infty}$ is, in particular, Hurewicz. Products of Hurewicz sets and $\mathfrak{d}$-concentrated real sets are Menger [30, Theorem 4.6]. $\quad \square$

The set $X \cup [\mathbb{N}]^{<\infty}$ in Theorem 4.7 is not Hurewicz since its image under the function $x \mapsto x^{\mathsf{c}}$ is unbounded in $[\mathbb{N}]^\infty$. The existence of non-Hurewicz sets of this form follows from a weaker hypothesis [29, Theorem 3.9], but without the non-productive property. The product of every Hurewicz real set and every $\mathfrak{d}$-concentrated real set is Menger [30, Theorem 4.6]. The following theorem implies that this assertion cannot be established for spaces that are not Hurewicz.

Let P be a property of topological spaces. A real set X is *productively P* if for each topological space Y with the property P, the product space $X \times Y$ has the property P. The question whether productively P implies productively Q, for P and Q covering properties among those studied here, has a long history. The remainder of this paragraph assumes that $\mathfrak{d} = \aleph_1$. Aurichi and Tall [3] improved several earlier results by proving that every productively Lindelöf space is Hurewicz. It was later shown that every productively Lindelöf space is productively Hurewicz and productively Menger [18, Theorem 8.2]. Thus, productively Lindelöf implies productively Menger, and the following theorem shows that productively Menger suffices to imply productively Hurewicz.

Theorem 4.8. *Assume that $\mathfrak{b} = \mathfrak{d}$.*

(1) *For every unbounded set $X \subseteq [\mathbb{N}]^\infty$, there is a $\mathfrak{d}$-concentrated real set Y such that the planar set $X \times Y$ is not Menger.*
(2) *In the realm of hereditarily Lindelöf spaces: If a real set X is productively Menger, then it is productively Hurewicz.*

Proof. (1) Let $\{ d_\alpha : \alpha < \mathfrak{d} \}$ be a dominating set in $[\mathbb{N}]^\infty$. Since $\mathfrak{b} = \mathfrak{d}$, for each ordinal number $\alpha < \mathfrak{d}$ the set $\{ d_\beta : \beta < \alpha \}$ is bounded, and thus there is a function $x_\alpha \in X$ such that $\{ d_\beta, x_\beta : \beta < \alpha \} <^\infty x_\alpha$. Then the subset $\{ x_\alpha : \alpha < \mathfrak{d} \}$ of the set X is $\mathfrak{d}$-unbounded, and Theorem 2.7 applies.

(2) Assume that there is a Hurewicz hereditarily Lindelöf space H such that the product space $X \times H$ is not Hurewicz. Then there is an unbounded upper continuous image Z of the space $X \times H$ in $[\mathbb{N}]^\infty$ [18, Theorem 7.3]. By (1), there is a $\mathfrak{d}$-concentrated real set Y such that the planar set $Z \times Y$ is not Menger. Since the set $Z \times Y$ is an upper continuous image of the product space $X \times H \times Y$, the latter space is not

Menger, too. As the space H is Hurewicz and hereditarily Lindelöf and the set Y is $\mathfrak{d}$-concentrated, the product space $H \times Y$ is Menger [30, Theorem 4.6]. In summary, the product of the set X and the Menger, hereditarily Lindelöf space $H \times Y$ is not Menger. $\quad\square$

Some special hypothesis is necessary for Theorem 4.8: The union of less than $\mathfrak{g}$ Menger real sets is Menger [31,28]. Assume that $\mathfrak{b} < \mathfrak{g}$. Then any unbounded real set $X \subseteq [\mathbb{N}]^\infty$ of cardinality $\mathfrak{b}$ is productively Menger but not Hurewicz.

5. Productive real sets

In this section, we establish preservation of some properties under products. We begin with a generalization of an earlier result [18, Lemma 6.3] to general topological spaces. The earlier proof [18, Lemma 6.3] does not apply in this general setting; we provide an alternative proof.

Lemma 5.1 *(Productive Two Worlds Lemma). Let X be a subset of $[\mathbb{N}]^\infty$, Y be an arbitrary space, and $\Psi\colon (X \cup [\mathbb{N}]^{<\infty}) \times Y \to [\mathbb{N}]^\infty$ be an upper continuous function. There is an upper continuous function $\Phi\colon Y \to [\mathbb{N}]^\infty$ such that, for all points $x \in X$ and $y \in Y$, and all natural numbers n:*

$$\text{If } \Phi(y)(n) \leq x(n), \text{ then } \Psi(x, y)(n) \leq \Phi(y)(n).$$

Proof. For natural numbers n and m, let $U_m^n := \Psi^{-1}[\{\, a \in [\mathbb{N}]^\infty : a(n) \leq m \,\}]$. For each natural number n, the family $\{\, U_m^n : m \in \mathbb{N} \,\}$ is an ascending open cover of the product space $(X \cup [\mathbb{N}]^{<\infty}) \times Y$. By enlarging the sets U_m^n, we may assume that they are open in the larger space $\mathrm{P}(\mathbb{N}) \times Y$.

Fix a point $y \in Y$ and a natural number n. Set $a_y(1) := 1$ and $V_1^y := Y$. For a natural number k, let m be the minimal natural number with $\mathrm{P}([1, a_y(k))) \times \{y\} \subseteq U_m^n$. Let $a_y(k+1)$ be the minimal natural number such that $a_y(k+1) \geq m$ and $\mathrm{P}([a_y(k), a_y(k+1))^{\mathrm{c}}) \times \{y\} \subseteq U_m^n$. Since our open covers are ascending, we have

$$\mathrm{P}([a_y(k), a_y(k+1))^{\mathrm{c}}) \times \{y\} \subseteq U_{a_y(k+1)}^n.$$

Notice that the number $a_y(k+1)$ is minimal with this property. As the set $\mathrm{P}([a_y(k), a_y(k+1))^{\mathrm{c}}) \times \{y\}$ is compact, there is an open neighborhood $V_{k+1}^y \subseteq V_k^y$ of the point y such that

$$\mathrm{P}([a_y(k), a_y(k+1))^{\mathrm{c}}) \times V_{k+1}^y \subseteq U_{a_y(k+1)}^n.$$

Define

$$\Phi(y)(n) := a_y(n+1).$$

For each point $y' \in V_{n+1}^y$, we have $y' \in V_k^y$ for all $k = 1, \dots, n+1$. The sequence $a_{y'}(1), \dots, a_{y'}(n+1)$ is bounded by the sequence $a_y(1), \dots, a_y(n+1)$, coordinate-wise: For $k = 1$, we have $a_{y'}(1) = a_y(1) = 1$. Assume that $a_{y'}(k) \leq a_y(k)$. Then

$$\mathrm{P}([a_{y'}(k), a_y(k+1))^{\mathrm{c}}) \times \{y'\} \subseteq \mathrm{P}([a_y(k), a_y(k+1))^{\mathrm{c}}) \times V_{k+1}^y \subseteq U_{a_y(k+1)}^n,$$

and, by the minimality of the number $a_{y'}(k+1)$, we have $a_{y'}(k+1) \leq a_y(k+1)$, too.

In summary, we have $\Phi(y')(n) = a_{y'}(n+1) \leq a_y(n+1) \leq \Phi(y)(n)$ for all points $y' \in V_{n+1}^y$. This shows that the function Φ is upper continuous.

Fix a point $y \in Y$ and a natural number n. Let $x \in [\mathbb{N}]^\infty$ be an element with $\Phi(y)(n) \leq x(n)$. As $a_y(n+1) = \Phi(y)(n) \leq x(n)$, there is a natural number $k \leq n$ such that $x \cap [a_y(k), a_y(k+1)) = \emptyset$. Thus,

$$(x, y) \in \mathrm{P}([a_y(k), a_y(k+1))^c) \times \{y\} \subseteq U^n_{a_y(k+1)} \subseteq U^n_{\Phi(y)(n)},$$

and therefore $\Psi(x, y)(n) \leq \Phi(y)(n)$. $\square$

For filters, we obtain a productive version of Proposition 4.5.

Theorem 5.2. *Let F be a filter and $X \subseteq [\mathbb{N}]^\infty$ be an F-scale. For each F-Menger space Y, every upper continuous image of the product space $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ in $[\mathbb{N}]^\infty$ is $\leq_{F_a}$-bounded for some set $a \in F^+$.*

Proof. Let $\Psi \colon (X \cup [\mathbb{N}]^{<\infty}) \times Y \to [\mathbb{N}]^\infty$ be an upper continuous function. Let $\Phi \colon Y \to [\mathbb{N}]^\infty$ be as in the Productive Two Worlds Lemma (Lemma 5.1). Since the space Y is F-Menger, there is a function $b \in [\mathbb{N}]^\infty$ such that $\Phi[Y] \leq_F b$. As the set X is an F-scale, there is a function $c \in [\mathbb{N}]^\infty$ such that $b \leq_{F^+} c \leq_F x$ for all but less than $\mathfrak{b}(F)$ elements of X. Let $a := [b \leq c]$, an element of the semifilter F^+. Then the cardinality of the set

$$Z := \{\, x \in X : b \not\leq_{F_a} x \,\}$$

is smaller than $\mathfrak{b}(F)$.

Fix a pair $(x, y) \in (X \setminus Z) \times Y$. Then $b \leq_{F_a} x$ and $\Phi(y) \leq_F b$. Since F is a filter, we have

$$[\Phi(y) \leq x] \supseteq [\Phi(y) \leq b] \cap [b \leq x] \in F_a.$$

This shows that $\Psi[(X \setminus Z) \times Y] \leq_{F_a} b$. Let $z \in Z \cup [\mathbb{N}]^{<\infty}$. Since the set $\{z\} \times Y$ is F-Menger, $\Psi[\{z\} \times Y] \leq_F c_z$ for some function $c_z \in [\mathbb{N}]^\infty$. Since $|Z \cup [\mathbb{N}]^{<\infty}| < \mathfrak{b}(F)$, there is a function $c \in [\mathbb{N}]^\infty$ such that $\{\, c_z : z \in Z \cup [\mathbb{N}]^{<\infty} \,\} \leq_F c$. As F is a filter, we have $\Psi[(Z \cup [\mathbb{N}]^{<\infty}) \times Y] \leq_F c$, and therefore $\Psi[(X \cup [\mathbb{N}]^{<\infty}) \times Y] \leq_{F_a} \{\, \max\{b(n), c(n)\} : n \in \mathbb{N} \,\}$. $\square$

Theorem 5.3. *Let F be a filter, and $X \subseteq [\mathbb{N}]^\infty$ be an F-scale. In the realm of hereditarily Lindelöf spaces:*

(1) *For each F-Menger space Y, the product space $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ is F^+-Menger.*
(2) *If F is an ultrafilter, then the real set $X \cup [\mathbb{N}]^{<\infty}$ is productively F-Menger.*

Proof. Every product of a metrizable Lindelöf space and a hereditarily Lindelöf space is Lindelöf. Apply Theorem 5.2. $\square$

The following theorem was previously known for $\mathfrak{b}$-scales, a special kind of cF-scales [18, Theorem 6.5]. This theorem and the subsequent one improve upon earlier results [29, Corollary 4.4], asserting that the corresponding properties hold in all finite powers.

A semifilter S is meager if and only if there is a function $h \in [\mathbb{N}]^\infty$ such that for each set $s \in S$, the set $s \cap [h(n), h(n+1))$ is nonempty for almost all natural numbers n [23, Theorem 21]. For meager semifilters S, we have $\mathfrak{b}(S) = \mathfrak{b}$ [29, Corollary 2.27], and S-Menger is equivalent to Hurewicz [29, Theorem 2.32]. The following theorem generalizes an earlier result [29, Theorem 2.28], using a similar proof.

Theorem 5.4. *Let S be a meager semifilter, and $X \subseteq [\mathbb{N}]^\infty$ be an S-scale. Then, in the realm of hereditarily Lindelöf spaces, the real set $X \cup [\mathbb{N}]^{<\infty}$ is productively Hurewicz.*

Proof. Let Y be a hereditarily Lindelöf, Hurewicz space. Since the space Y is hereditarily Lindelöf, the product space $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ is Lindelöf. Let $\Phi \colon X \times Y \to [\mathbb{N}]^\infty$ be an upper continuous function and $\Psi \colon Y \to [\mathbb{N}]^\infty$ be the upper continuous function provided by the Productive Two Worlds Lemma

(Lemma 5.1). Since the space Y is Hurewicz, its image $\Psi[Y]$ is $\leq^*$-bounded by some function $b \in [\mathbb{N}]^\infty$. Let $h \in [\mathbb{N}]^\infty$ be a witness for the semifilter S being meager. Define a function $\tilde{b} \in \mathbb{N}^\mathbb{N}$ by

$$\tilde{b}(k) := b(h(n+2))$$

for all natural numbers n and for $k \in [h(n), h(n+1))$. Then $\Psi[Y] \leq^* b \leq \tilde{b}$.

Since the set X is an S-scale, there is a function $c \in [\mathbb{N}]^\infty$ such that $\tilde{b} \leq_{S+} c$ and all but less than $\mathfrak{b}$ functions $x \in X$ belong to the set

$$\tilde{X} := \{\, x \in X : c \leq_S x \,\}.$$

Claim 5.5. *The set $\Phi[(\tilde{X} \cup [\mathbb{N}]^{<\infty}) \times Y]$ is $\leq^*$-bounded.*

Proof. Fix a function $x \in \tilde{X}$. Then $[c \leq x] \in S$, and thus the set $[c \leq x] \cap [h(n), h(n+1))$ is nonempty for almost all natural numbers n. Let

$$d := \{\, n \in \mathbb{N} : [\tilde{b} \leq c] \cap [h(n-1), h(n)) \neq \emptyset \,\}.$$

Then, for almost all natural numbers $n \in d$, there are natural numbers $l \in [\tilde{b} \leq c] \cap [h(n-1), h(n))$ and $m \in [c \leq x] \cap [h(n), h(n+1))$, and we have

$$b(h(n+1)) = \tilde{b}(l) \leq c(l) \leq c(m) \leq x(m) \leq x(h(n+1)).$$

Thus, $b(k) \leq x(k)$ for almost all natural numbers $k \in e := \{\, h(n+1) : n \in d \,\}$.

Let $y \in Y$. Since $\Psi[Y] \leq^* b$, for almost all natural numbers $k \in e$ we have

$$\Psi(y)(k) \leq b(k) \leq x(k),$$

and thus

$$\Phi(x,y)(k) \leq \Psi(y)(k) \leq b(k).$$

Hence, the set $\Phi[\tilde{X} \times Y]$ is $\leq^*$-bounded on an infinite set, and thus [10, Fact 3.4] $\leq^*$-bounded.   □

As $|(X \setminus \tilde{X}) \cup [\mathbb{N}]^{<\infty}| < \mathfrak{b}$ and the space Y is Hurewicz, the image $\Phi[((X \setminus \tilde{X}) \cup [\mathbb{N}]^{<\infty}) \times Y]$ is a union of less than $\mathfrak{b}$ sets that are $\leq^*$-bounded, and is thus $\leq^*$-bounded. Thus, the entire image $\Phi[(X \cup [\mathbb{N}]^{<\infty}) \times Y]$ is $\leq^*$-bounded.   □

6. Cofinal S-scales

For a semifilter S, the following special type of S-scale is a natural generalization of the earlier notion of cofinal $\mathfrak{b}(S)$-scale [29, Definition 2.22].

Definition 6.1. Let S be a semifilter. A set $X \subseteq [\mathbb{N}]^\infty$ with $|X| \geq \mathfrak{b}(S)$ is a *cofinal S-scale* if for each function $b \in [\mathbb{N}]^\infty$, we have

$$b \leq_S x$$

for all but less than $\mathfrak{b}(S)$ functions $x \in X$.

For example, a set $X \subseteq [\mathbb{N}]^\infty$ is a cofinal $[\mathbb{N}]^\infty$-scale if and only if the set X is $\mathfrak{d}$-unbounded. Thus, for some semifilters S, cofinal S-scales provably exist. But this is not always the case.

Proposition 6.2. *Let S be a semifilter.*

(1) *Cofinal S-scales are $\leq_{S^+}$-unbounded.*
(2) *Every subset, of cardinality $\mathfrak{b}(S)$, of a cofinal S-scale is a cofinal S-scale.*
(3) *If there is a cofinal S-scale, then $\mathfrak{b}(S^+) \leq \mathfrak{b}(S)$.*
(4) *If $\mathfrak{b}(S) = \mathfrak{d}$, then there is a cofinal S-scale [29, Lemma 2.23].* □

Corollary 6.3. *Let F be a filter. The following assertions are equivalent:*

(1) *There is a cofinal F-scale.*
(2) *$\mathfrak{b}(F) = \mathfrak{b}(F^+)$.*

Proof. $(1) \Rightarrow (2)$: Since F is a filter, we have $F \subseteq F^+$, and thus $\mathfrak{b}(F) \leq \mathfrak{b}(F^+)$. Apply Proposition 6.2(3).

$(2) \Rightarrow (1)$: A $\leq_{F^+}$-unbounded set $\{\, b_\alpha : \alpha < \mathfrak{b}(F)\,\} \subseteq [\mathbb{N}]^\infty$ is $\leq_F$-cofinal. For each ordinal number $\alpha < \mathfrak{b}(F)$, let $x_\alpha \in [\mathbb{N}]^\infty$ be such that $\{\, b_\beta, x_\beta : \beta < \alpha \,\} \leq_F x_\alpha$. As F is a filter, the relation $\leq_F$ is transitive, and thus the set $\{\, x_\alpha : \alpha < \mathfrak{b}(F)\,\}$ is a cofinal F-scale. □

In particular, since $\mathfrak{b}(\mathrm{cF}) = \mathfrak{b}$ and $\mathfrak{b}(\mathrm{cF}^+) = \mathfrak{b}([\mathbb{N}]^\infty) = \mathfrak{d}$, there are cofinal cF-scales if and only if $\mathfrak{b} = \mathfrak{d}$. The proof of the following theorem is similar to that of Theorems 5.2–5.3(1).

Theorem 6.4. *Let F be a filter and $X \subseteq [\mathbb{N}]^\infty$ be a cofinal F-scale. Then, in the realm of hereditarily Lindelöf spaces, the real set $X \cup [\mathbb{N}]^{<\infty}$ is productively F-Menger.* □

Theorem 6.5. *Let $X \subseteq [\mathbb{N}]^\infty$ be a cofinal cF-scale. Then the real set $X \cup [\mathbb{N}]^{<\infty}$ is productively S-Menger for all semifilters S.*

Proof. We modify the proof of Theorem 5.2. Let Y be an S-Menger space, and $\Psi \colon (X \cup [\mathbb{N}]^{<\infty}) \times Y \to [\mathbb{N}]^\infty$ be an upper continuous function. Let $\Phi \colon Y \to [\mathbb{N}]^\infty$ be as in the Productive Two Worlds Lemma (Lemma 5.1). Since the space Y is S-Menger, there is a function $b \in [\mathbb{N}]^\infty$ such that $\Phi[Y] \leq_S b$. As the set X is a cofinal cF-scale, the cardinality of the set

$$Z := \{\, x \in X : b \not\leq^* x \,\}$$

is smaller than $\mathfrak{b}$.

Fix a pair $(x, y) \in (X \setminus Z) \times Y$. Then $b \leq^* x$ and $[\Phi(y) \leq b] \in S$. Thus, for almost all natural numbers $n \in [\Phi(y) \leq b]$, we have $\Phi(y)(n) \leq b(n) \leq x(n)$, and therefore

$$\Psi(x, y)(n) \leq \Phi(y)(n) \leq b(n).$$

Since semifilters are invariant under finite modifications of their elements, we have $\Psi(x, y) \leq_S b$. This shows that $\Psi[(X \setminus Z) \times Y] \leq_S b$.

The remainder of the proof is identical to that of Theorem 5.2. □

A *superfilter* (also called *grille* or *coideal*) is a semifilter S such that $a \cup b \in S$ implies $a \in S$ or $b \in S$. A semifilter S is a superfilter if and only if the semifilter S^+ is a filter. Equivalently, a superfilter is a union of a family of ultrafilters. For example, the set $[\mathbb{N}]^\infty = \mathrm{cF}^+$ is a superfilter.

Proposition 6.6. *Let S be a superfilter. A set $X \subseteq [\mathbb{N}]^\infty$ is a cofinal S-scale if and only if it is an S-scale.*

Proof ($\Leftarrow$). Let $F := S^+$. If $b \leq_{S^+} c \leq_S x$, then $b \leq_F c \leq_{F^+} x$, and since F is a filter, we have $b \leq_{F^+} x$, that is, $b \leq_S x$. $\quad\square$

The proof of the following assertion is similar to that of Proposition 4.5.

Proposition 6.7. *Let S be a semifilter. For each cofinal S-scale X, the real set $X \cup [\mathbb{N}]^{<\infty}$ is S-Menger.* $\quad\square$

Let U be an ultrafilter, and $X \subseteq [\mathbb{N}]^\infty$ be a U-scale. By Proposition 6.6, the set X is in fact a cofinal U-scale. Using Proposition 6.7, we obtain an alternative derivation of Corollary 4.6. Similarly, Theorem 6.4 generalizes Theorem 5.3(2).

Theorem 6.4 cannot be extended to all semifilters, and not even to all superfilters: By Theorems 3.2–3.3, the hypothesis $\mathfrak{d} \leq \mathfrak{r}$ implies that Theorem 6.4 does not hold for the superfilter $[\mathbb{N}]^\infty$. The latter assertion also follows from the following theorem that is, in fact, established by the proof of Theorem 4.7.

Theorem 6.8. *Assume that $\mathfrak{b} = \mathfrak{d}$. Let S be a semifilter. The following assertions are equivalent:*

(1) *The semifilter S is nonmeager.*
(2) *There are an S-scale $X \subseteq [\mathbb{N}]^\infty$ and a $\mathfrak{d}$-concentrated real set Y such that the planar set $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ is not Menger.*
(3) *There are a cofinal S-scale $X \subseteq [\mathbb{N}]^\infty$ and a $\mathfrak{d}$-concentrated real set Y such that the planar set $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ is not Menger.* $\quad\square$

A related result of Repovš and Zdomskyy [19, Theorem 3.3] asserts that, if $\mathfrak{b} = \mathfrak{d}$, then there are ultrafilters U_1 and U_2, a (cofinal) U_1-scale X_1, and a (cofinal) U_2-scale X_2, such that the planar set $(X_1 \cup [\mathbb{N}]^{<\infty}) \times (X_2 \cup [\mathbb{N}]^{<\infty})$ is not Menger.

Theorem 6.9. *Assume that $\mathfrak{b} = \mathfrak{d}$. For every nonmeager filter F:*

(1) *In the realm of hereditarily Lindelöf spaces, there is a productively F-Menger space that is not Hurewicz and not productively Menger.*
(2) *The property F-Menger is strictly between Hurewicz and Menger.*

Proof of (1). By Theorem 6.8(3) and Theorem 6.4, using that products of Hurewicz sets and $\mathfrak{d}$-concentrated real sets are Menger [30, Theorem 4.6]. $\quad\square$

7. Comments and open problems

Except for the last subsection of this section, we restrict attention to real sets throughout this section. Thus, we omit the adjective *real* almost throughout this section.

The *Menger productivity problem*, whether Menger's property is consistently preserved by finite products, remains open. The hypothesis $\mathfrak{d} \leq \mathfrak{r}$ provides two Menger sets whose product is not Menger (Theorems 2.7 and 3.2). It is well known that this immediately provides a Menger set whose square is not Menger. Indeed, assume that X and Y are Menger sets such that the planar set $X \times Y$ is not Menger. The set $X \cup Y$ is Menger. We may assume that $X \subseteq (0,1)$ and $Y \subseteq (2,3)$. Then the set $X \times Y$ is a closed subset of the square $(X \cup Y)^2$. Menger's property is hereditary for closed subsets.

7.1. A combinatorial characterization of the cardinal number $\min\{\mathfrak{r}, \mathfrak{d}\}$

Aubrey [1] proved that $\min\{\mathfrak{d}, \mathfrak{u}\} \leq \mathfrak{r}$. Since $\mathfrak{r} \leq \mathfrak{u}$, the hypothesis $\mathfrak{d} \leq \mathfrak{r}$ in Theorem 3.3 is equivalent to the hypothesis $\mathfrak{d} \leq \mathfrak{u}$.

Initially, we proved Theorem 3.3 using a new hypothesis.

Definition 7.1. Let $\mathfrak{bidi}$ be the minimal cardinality of a set $X \subseteq [\mathbb{N}]^{\infty}$ such that there is no set $b \in [\mathbb{N}]^{\infty,\infty}$ with $X \leq^{\infty} b, b^{\mathfrak{c}}$.

We observed that $\max\{\mathfrak{b}, \mathrm{cov}(\mathcal{M})\} \leq \mathfrak{bidi} \leq \min\{\mathfrak{r}, \mathfrak{d}\}$, and needed that $\mathfrak{bidi} = \mathfrak{d}$ to carry out our construction. It is immediate that $\mathfrak{bidi} \leq \mathfrak{d}$, and the argument in the proof of the implication $(2) \Rightarrow (1)$ in Theorem 3.3 shows that $\mathfrak{bidi} \leq \mathfrak{r}$. Answering our question, Mejía pointed out to us that, by a result of Kamburelis and Węglorz [13], our upper bound on the cardinal number $\mathfrak{bidi}$ is tight [16] (see Lemma 3.4). We thus have the following characterization of $\min\{\mathfrak{r}, \mathfrak{d}\}$.

Proposition 7.2. $\mathfrak{bidi} = \min\{\mathfrak{r}, \mathfrak{d}\}$. $\square$

7.2. Which κ-unbounded sets are not productively Menger?

There are (e.g., by Proposition 4.2), in ZFC, $\mathfrak{b}$-unbounded sets. Every union of less than $\max\{\mathfrak{b}, \mathfrak{g}\}$ Menger sets is Menger [31,28]. Since the hypothesis $\mathfrak{b} < \mathfrak{g}$ is consistent, Theorem 2.7 and Corollary 2.11 do not extend to the case $\kappa = \mathfrak{b}$, or to any cardinal number that is consistently smaller than $\max\{\mathfrak{b}, \mathfrak{g}\}$.

For a κ-unbounded set, which we may assume to have cardinality κ, the present proof of Theorem 2.7 requires a partition $\mathfrak{d} = \bigcup_{\alpha < \kappa} I_{\alpha}$ such that for each set $J \subseteq \kappa$ with $|J| < \mathfrak{d}$, we have $\left| \bigcup_{\alpha \in J} I_{\alpha} \right| < \mathfrak{d}$. It is not difficult to see that this implies that $\kappa \in \{\mathrm{cf}(\mathfrak{d}), \mathfrak{d}\}$.

Problem 7.3. Assume that κ is a cardinal number with $\mathrm{cf}(\mathfrak{d}) < \kappa < \mathfrak{d}$. Let X be a κ-unbounded set in $[\mathbb{N}]^{\infty}$. Is there necessarily a $\mathfrak{d}$-concentrated set Y such that the planar set $X \times Y$ is not Menger?

7.3. Products of Hurewicz sets

Scheepers [25, Problem 6.7] has also asked whether, *consistently*, every product of two Hurewicz sets is Hurewicz. It seems that a combination of the methods of this paper and ones from a recent paper of Repovš and Zdomskyy [20] yields a positive answer. We plan to carry out this program, together with Zdomskyy, in a sequel paper.

Problem 7.4. Find mild hypotheses implying that there are two Hurewicz sets whose product is not Hurewicz.

In the notation of Section 4, Menger's property is $[\mathbb{N}]^{\infty}$-Menger, and Hurewicz's property is cF-Menger. By Theorems 3.2 and 3.3, $[\mathbb{N}]^{\infty}$-scales need not be productively $[\mathbb{N}]^{\infty}$-Menger. In contrast, by Theorem 5.4, cF-scales are productively cF-Menger. Thus, modern constructions of Hurewicz sets do not help in regard to Problem 7.4.

The classic example of a Hurewicz set (if not counting σ-compact sets, which are productively Hurewicz) is a Sierpiński set [26]. If $\mathfrak{b} = \mathrm{cov}(\mathcal{N}) = \mathrm{cof}(\mathcal{N})$, then there is a $\mathfrak{b}$-Sierpiński set (which is Hurewicz) whose square is not Hurewicz [21, Theorem 43]. No more general constructions are known.

Problem 7.5. Does the Continuum Hypothesis imply that no Sierpiński set is productively Hurewicz? Productively Menger?

By Theorem 4.8, if $\mathfrak{b} = \mathfrak{d}$, then every productively Menger set is productively Hurewicz. By the discussion following that theorem, if $\mathfrak{b} < \mathfrak{g}$ then there are productively Menger sets that are not Hurewicz.

Problem 7.6. Assume the Continuum Hypothesis. Do the classes of productively Menger and productively Hurewicz sets coincide?

Recall that for meager semifilters S, being S-Menger is equivalent to being Hurewicz [29, Theorem 2.32]. By Theorem 5.4, in this case, for each S-scale X, the set $X \cup [\mathbb{N}]^{<\infty}$ is productively S-Menger.

Problem 7.7. Assume the Continuum Hypothesis. For which semifilters S there is an S-scale X such that the set $X \cup [\mathbb{N}]^{<\infty}$ is not productively S-Menger?

Thus meager semifilters do not have the property in Problem 7.7. By Theorem 6.4 and Proposition 6.6, ultrafilters are also not in that category. But the full semifilter $[\mathbb{N}]^\infty$ is in this category, by Theorem 6.8.

A $\mathfrak{b}$-scale [26] is a particularly simple kind of a cF-scale, for cF the filter of cofinite sets.

Problem 7.8. Let $X \subseteq [\mathbb{N}]^\infty$ be a $\mathfrak{b}$-scale. Is the set $X \cup [\mathbb{N}]^{<\infty}$ necessarily productively Menger?

If $\mathfrak{u} < \mathfrak{g}$, then every $\mathfrak{d}$-concentrated set (in particular, every union of an S-scale, for some semifilter S, and the set $[\mathbb{N}]^{<\infty}$) is productively Menger [18, Theorem 4.7].

7.4. Strictly unbounded sets

Say that a set $X \subseteq [\mathbb{N}]^\infty$ is *strictly unbounded* if for every set $A \subseteq [\mathbb{N}]^\infty$ of cardinality smaller than $\mathfrak{d}$ there is a function $x \in X$ such that $A \leq^\infty x$. Let $X \subseteq [\mathbb{N}]^\infty$ be a strictly unbounded set. By the argument in the proof of Theorem 4.8(1), the set X contains a $\mathfrak{d}$-unbounded set. By Theorem 2.7, there is a $\mathfrak{d}$-concentrated set Y such that the planar set $X \times Y$ is not Menger. If $\mathfrak{b} = \mathfrak{d}$, then every unbounded set in $[\mathbb{N}]^\infty$ is strictly unbounded. We thus obtain a generalization of Theorem 4.8.

The construction in Theorem 3.3 that provides a Menger set that is not productively Menger provides, in fact, a Menger strictly unbounded set.

A negative answer to the second item of the following problem implies a negative solution for the Menger productivity problem.

Problem 7.9.

(1) Is it consistent that $\mathfrak{r} < \mathfrak{d}$ and there are strictly unbounded Menger sets?
(2) Is it consistent that there are no strictly unbounded Menger sets?

7.5. General spaces

Let S be a semifilter. Restricting the definition of S-*Menger* spaces to countable open covers, we obtain the definition of *countably S-Menger* spaces. This makes it possible to eliminate the adjective "hereditarily Lindelöf" in most of our theorems.

For *general* Hurewicz spaces, the following problem remains open, even for the so-called $\mathfrak{b}$-scales [26, Definition 2.8].

Problem 7.10. Let cF be the semifilter of cofinite sets, $X \subseteq [\mathbb{N}]^\infty$ be an cF-scale, and Y be a Hurewicz space. Is the product space $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ necessarily Hurewicz?

Theorem 5.4 provides a positive answer for hereditarily Lindelöf spaces Y. But this restriction is only needed for deducing that the product space $(X \cup [\mathbb{N}]^{<\infty}) \times Y$ is Lindelöf (and similarly for the other results in Section 4). A positive solution for the following problem would suffice.

Problem 7.11. Let X be a real set of cardinality smaller than $\mathfrak{b}$, and Y be a Hurewicz space. Is the product space $X \times Y$ necessarily Lindelöf?

We consider productive properties of general S-Menger spaces in a sequel paper [22]. There, the results have a somewhat different character.

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